Content:

Free, Damped and Forced Vibration:

Simple Harmonic motion (SHM), differential equation for SHM (No derivation), spring mass and its applications.

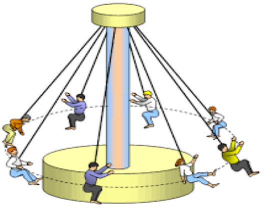
Theory of damped oscillations (Derivation), Types of damping (Graphical Approach). Engineering applications of damped oscillations, Theory of forced oscillations (Qualitative), resonance and sharpness of resonance, Numerical problems.

Introduction:

Motion of bodies can be broadly classified into three categories:

[1] Translational motion
[2] Rotational motion
[3] Vibrational / Oscillatory motion

Translational motion: When the position of a body varies linearly with time, such a motion is termed as translational motion. Example: A car moving on a straight road, a ball moving on the ground.



Rotational motion: When a body as a whole does not change its position linearly with time but rotates about its axis, this motion is said to be rotational motion. Example: rotation of a fly wheel on ball bearings.

Vibrational /Oscillatory motion: When a body executes back and forth motion which repeats over and again about a mean position, then the body is said to have Vibrational/oscillatory motion. If such motion repeats in regular intervals of time then it is called *Periodic* *motion or Harmonic motion* and the body executing such motion is called **Harmonic oscillator.**

In harmonic motion there is a linear relation between force acting on the body and displacement produced. Example: bob of a pendulum clock, motion of prongs of tuning fork, motion of balance wheel of a watch, the up and down motion of a mass attached to a spring.

Note:

1. *If there is no linear relation between the force and displacement then the motion is called un harmonic motion. Many systems are un harmonic in nature.*
2. *In some oscillatory systems the bodies may be at rest but the physical properties of the system may undergo changes in oscillatory manner Examples: Variation of pressure in sound waves, variation of electric and magnetic fields in electromagnetic waves.*

Parameters of an oscillatory system

1. **Mean position**: The position of the oscillating body at rest.

2. **Amplitude:** The amplitude of an SHM is the maximum displacement of the body from its mean position.

3. **Time Period:** The time interval during which the oscillation repeats itself is called the time period. It is denoted by **T** and its unit in seconds.

$$\text{Period} = T = 2\pi \sqrt{\frac{\text{Displacement}}{\text{Acceleration}}}$$

4. **Frequency:** The number of oscillations that a body completes in one second is called the frequency of periodic motion. It is the reciprocal of the time period T and it is denoted by f

   $$\text{Frequency } f=1/T$$

5. **Angular frequency:** This is the orbital frequency or circular frequency and it is measured in radians per second. It is denoted by ω

   $$\omega = 2\pi \text{ x } 1/T = 2\pi f$$

6. **Phase:** It is the stateof an oscillating system. If the SHM is represented by $y=A\sin(\omega t+\phi)$, together with ω as the angular frequency. The quantity $(\omega t+\phi)$ of the sine function is called the total phase of the motion at time 't' and 'ϕ' is the initial phase or epoch.

All periodic motions are not vibratory or oscillatory. In this chapter we shall study the simplest vibratory motion along one dimension called **Simple Harmonic Motion** (SHM)

SIMPLE HARMONIC MOTION (SHM)

A body is said to be undergoing Simple Harmonic Motion (SHM) when the acceleration of the body is always proportional to its displacement and is directed towards its equilibrium or mean position. Simple harmonic motion can be broadly classified in to two types, namely **linear simple harmonic motion and angular simple harmonic motion.**

Linear Simple Harmonic Motion: If the body executing SHM has a linear acceleration then the motion of the body is linear simple harmonic motion.

Examples: motion of simple pendulum, the motion of a point mass tied with a spring etc.,

Angular Simple Harmonic Motion: If the body executing SHM has an angular acceleration then the motion of the body is angular simple harmonic motion.

Examples: Oscillations of a torsional pendulum

Free, Damped and Forced Vibrations

A particle or a system which executes simple harmonic motion is called ***Simple Harmonic Oscillator***. Examples of simple harmonic motion are motion of the bob of a simple pendulum, motion of a point mass fastened to spring, motion of prongs of tuning fork etc.,

Free vibrations or un damped vibrations

If a body oscillates without the influence of any external force, then the oscillations are called **free oscillations or un damped oscillations**. In free oscillations the body oscillates with its natural frequency and the amplitude remains constant (Fig.1)

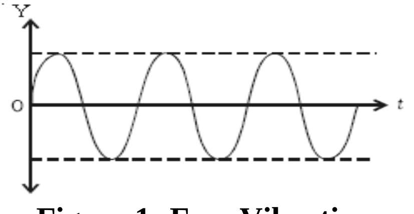


Figure 1: Free Vibrations

In practice it is not possible, actually the amplitude of the vibrating body decreases to zero as a result of friction. Hence, practical examples for free oscillation/vibrations are those in which the friction in the system is negligibly small.

Examples of Simple harmonic oscillator

a) Spring and Mass system

The spring and mass system is an example for linear simple harmonic oscillator.

The spring mass system consists of a block of mass *m* suspended from a rigid support by means of a mass less spring (ie a spring whose mass is negligible).

In a spring mass system if the suspended mass is pulled gently, the spring undergoes an elongation and it is said to be deformed. In the deformed state an internal force is developed and it is opposite to the external deforming force. At the equilibrium state the internal force and external force are equal in magnitude. When the external force is removed the internal force restores the mass to its initial position. Hence the internal force is called the restoring force. The restoring force produced in the spring obeys **Hooke's Law**.

According to Hooke's **Law "The restoring force produced in a system is proportional to the displacement".** When the mass is displaced through a distance 'y' then the restoring force (F) produced is $F \propto y$

$$F = -ky \text{---------} (1)$$

The constant *k* is called **force constant or spring constant or stiffness constant**. It is a measure of the stiffness of the spring. The negative sign in equation (1) indicates that the restoring force is opposite to the direction of the external force or in the direction of its equilibrium position.

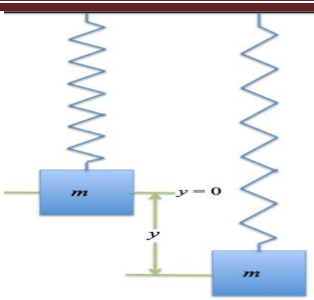


Figure 2: A one-dimensional simple harmonic oscillator.

Note: In equilibrium condition the linear restoring force (F) in magnitude is equal to the weight '*mg*' of the hanging mass.

i.e, F = mg $\quad\quad \therefore$ mg = ky

and spring constant $k = \frac{mg}{y}$ --------(2)

Now, if the mass is displaced down through a distance 'y' from its equilibrium position and released then it executes oscillatory motion.

Velocity of the body = $v = \frac{dy}{dt}$ and

Acceleration of the body = $a = \frac{d^2y}{d^2t}$

From Newton's second law of motion for the block we can write

$$F = ma = m\frac{d^2y}{dt^2} \text{-------------(3)}$$

from equation (1) substitue F= -ky in equation(3),

$$-ky = m\frac{d^2y}{dt^2}$$

or $\frac{d^2y}{dt^2} + \frac{k}{m}y = 0$,Substitue $\frac{k}{m} = \omega^2$,

Where ω is the angular natural frequency.

i.e., $\frac{d^2y}{dt^2} + \omega^2 y = 0$----------(4)

Equation (4) is the general differential equation for the *free oscillator*. The mathematical solution of the equation (4), y(t) represents the position as a function of time t. Let $y(t) = A\sin(\omega t + \phi)$ be the solution.

$$y(t) = A\sin(\omega t + \phi) \text{-------------------}(5)$$

Where A is the ***amplitude*** and $(\omega t + \phi)$ *is* called the ***phase***, ϕ is the initial phase (i.e., phase at t=0).

Period of the oscillator (T)

We have F = ma and also F = -ky. At equilibrium, magnitude of restoring force is equal to weight of hanging mass. i.e, ma = ky

$$\therefore \text{Acceleration} = a = \frac{ky}{m}$$

The period of the oscillator is

$$T = 2\pi\sqrt{\frac{\text{Displacement}}{\text{Acceleration}}} = 2\pi\sqrt{\frac{y}{a}} = 2\pi\sqrt{\frac{y}{\frac{ky}{m}}} = 2\pi\sqrt{\frac{m}{k}}$$

We know that $\omega = \sqrt{\frac{k}{m}}$ and $\frac{1}{\omega} = \sqrt{\frac{m}{k}}$, rewrite period expression

Time Period $T = \frac{2\pi}{\omega}$,

frequency $f = \frac{1}{T} = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$

b) Spring Constant in Series and Parallel combination

Springs in Series:

Consider two springs A and B of force constants K_1 and K_2 be connected in series and a mass m attached to the lower end of the bottom most spring as shown in the figure 3. When mass 'm' is pulled down a little and released. Let y be the displacement of mass 'm' at any instant of time. Let y_1 and y_2 be the extensions of two springs of A and B respectively.

Then total extension in the spring is

$$y = y_1 + y_2$$

In series the restoring force in each spring is same, so,

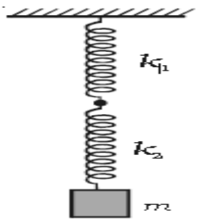


Figure 3: Springs in Series

$$F = -K_1 y_1 = -K_2 y_2$$

$$\Rightarrow \text{displacement } y_1 = -\frac{F}{K_1} \quad \text{and} \quad y_2 = -\frac{F}{K_2}$$

Total extension of the spring is $y = y_1 + y_2$

$$-\frac{F}{K_{eff}} = -\left[\frac{F}{K_1} + \frac{F}{K_2}\right]$$

Where K_{eff} is effective force constant of the spring

$$\therefore \quad \frac{1}{K_{eff}} = \frac{1}{K_1} + \frac{1}{K_2} = \frac{K_1 + K_2}{K_1 K_2}, \; K_{eff} = \frac{K_1 K_2}{K_1 + K_2}$$

So, time period of the body is given by

$$T = 2\pi\sqrt{\frac{m}{K_{eff}}} = 2\pi\sqrt{\frac{m(K_1 + K_2)}{K_1 K_2}}$$

Springs in Parallel:

Consider two springs A and B of force constants K_1 and K_2 be connected in parallel and a mass 'm' is attached at the lower end as shown in the figure 4. Let mass 'm' be pulled a little and released. Let 'y' be the displacement of mass 'm' from equilibrium position. Let F_1 and F_2 be the restoring forces developed in springs A and B respectively.

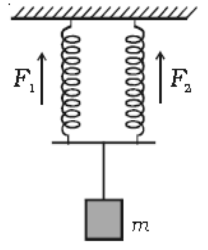


Figure 4: Springs in Parallel

Then net restoring force is, $F = F_1 + F_2$

Where $F_1 = -K_1 y$, $F_2 =- K_2 y$

$\therefore$ Total restoring force $F = (F_1 + F_2)$

$$-Ky = -(K_1 + K_2)\,y = K_{eff}\,y$$

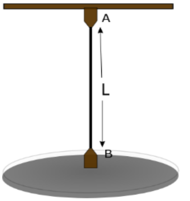
Where K_{eff} is effective force constant of spring

So, time period of the system is given by

$$T = 2\pi\sqrt{\frac{m}{K_{eff}}} = 2\pi\sqrt{\frac{m}{K_1 + K_2}}$$

Torsion Pendulum:

A pendulum in which the oscillations are due to the torsion (or twist) is a torsion pendulum.



A Torsional pendulum consists of a heavy metal disc suspended by means of a wire AB of length 'L'. The top end of the wire is fixed to a rigid support and the bottom end is fixed to the metal disc. When the disc is rotated in a horizontal plane so as to twist the wire, the various elements of the wire undergo deformation. The restoring couple in developed in the wire tries to bring the wire back to the original position. Therefore, disc executes torsional oscillations about the mean position. If θ is the angle of twist in the wire and 'C' is the couple per unit twist for the wire, then the restoring couple = Cθ.

At any instant, the deflecting couple (Iα) is equal to the restoring couple, (where I is the moment of inertia of the wire about the axis and α is the angular acceleration).

$$I\,\alpha = -\,C\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$\alpha = -\,(C/I)\theta \ldots\ldots\ldots\ldots\ldots(2)$$

The above relation shows that the angular acceleration is proportional to angular displacement and is always directed towards the mean position. The negative sign indicates that the restoring couple is in the opposite direction to the deflecting couple. Hence the system executes SHM Therefore; the time period of oscillator is given by relation.

$$\therefore Period\ T = 2\pi\sqrt{\frac{Displacement}{acceleration}} = 2\pi\sqrt{\frac{\theta}{(c/I)\theta}} = 2\pi\sqrt{\frac{I}{c}}$$

For a given wire of length l, radius r and rigidity modulus η the torsional constant c is given by.

$$\text{Torsional constant of the wire } C = \frac{\pi\eta r^4}{2l}$$

Damped vibrations:

Free vibrations are vibrations in which the friction/resistance considered is zero or negligible. Therefore, the body will keep on vibrating indefinitely with respect to time. In real sense if a body set into vibrations, its amplitude will be continuously decreasing due to friction/resistance and so the vibrations will die after some time, such vibrations are called ***damped vibrations***. "***<u>The vibrations which are subjected to external opposing force are called damped vibrations</u>***"

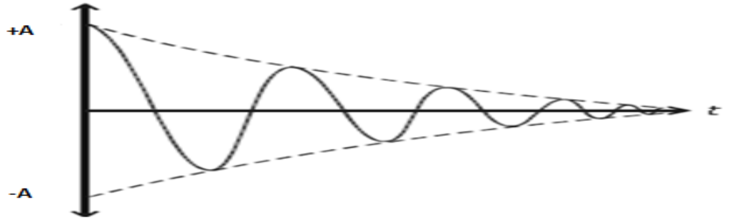
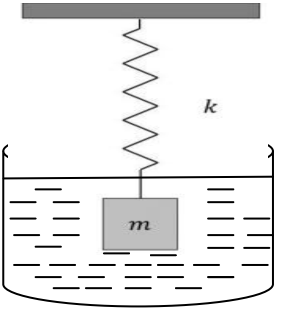


Figure 3: Damped vibrations

Differential equation of damped vibrations and solution

When a block of mass m suspended from a fixed end by means of a mass less spring in a viscous medium is set into vibrations the amplitude gradually decreases with time. As the mass vibrates in the liquid there will be a relative motion between the liquid and the mass and an opposing viscous force is developed. The viscous force which opposes the motion is force is proportional to the relative speed between the mass and the liquid.



Vertical spring and mass system

If 'y' is the displacement of the body from the equilibrium state at any instant of time't', then dy/dt is the instantaneous velocity and d^2y/dt^2 is the acceleration.

At any instant the two forces acting on the body are:

i. A restoring force which is proportional to displacement and acts in the opposite direction, it may be written as

$F_{restoring} = -ky$ *Where k is the spring constant*

ii. A *frictional or damping* force which is directly proportional to the velocity of the mass and is opposite to the motion, it may be written as

$F_{damping} = -r\dfrac{dy}{dt}$ where b is the damping constant

The net force acting on the oscillator is the summation of the two.

$$F = F_{restoring} + F_{damping} = -ky - r\frac{dy}{dt}$$

But by Newton's law of motion, $F = m\dfrac{d^2y}{dt^2}$ where m is the mass of the body and

$\dfrac{d^2y}{dt^2}$ is the accleration of the body.

$$\text{Then, } m\frac{d^2y}{dt^2} = -ky - r\frac{dy}{dt}$$

$$\text{or} \quad \frac{d^2y}{dt^2} + \frac{r}{m}\frac{dy}{dt} + \frac{k}{m}y = 0 \text{-----------(1)}$$

Let $\dfrac{r}{m} = 2b$ and $\dfrac{k}{m} = \omega^2$, then the above equation takes the form,

$$\frac{d^2y}{dt^2} + 2b\frac{dy}{dt} + \omega^2 y = 0 \text{ ------------------(2)}$$

[Note : Equation 2 reduces to the differential equation for free oscillations if b=0]

This is the differential equation of second order.

In order to solve this equation, we assume its solution as

$$y = Ae^{\alpha t} \text{ --------------------(3)}$$

Where A and α are 2 arbitrary constants (variational parameters).

Differentiating equation (3) with respect to time t, we have

$$\frac{dy}{dt} = A\alpha e^{\alpha t} \text{ and } \frac{d^2y}{dt^2} = A\alpha^2 e^{\alpha t}$$

By substituting these values in equation (2), we have

$$A\alpha^2 e^{\alpha t} + 2bA\alpha e^{\alpha t} + \omega^2 Ae^{\alpha t} = 0$$

or

$$Ae^{\alpha t}\left(\alpha^2 + 2b\alpha + \omega^2\right) = 0$$

For the above equation to be satisfied, either y=0, or

$$\left(\alpha^2 + 2b\alpha + \omega^2\right) = 0$$

Since y=0, corresponds to a trivial solution, one has to consider the solution

$$\left(\alpha^2 + 2b\alpha + \omega^2\right) = 0$$

The standard solution of the above quadratic equation has two roots, it is given by,

$$\alpha = \frac{-2b^2 \pm \sqrt{4b^2 - 4\omega^2}}{2}$$

$$\alpha = -b \pm \sqrt{b^2 - \omega^2}$$

Therefore, the general solution of equation (2) is given by

$$y = C\, e^{(-b + \sqrt{b^2 - \omega^2})t} + D\, e^{(-b - \sqrt{b^2 - \omega^2})t} \text{ -----(4)}$$

Where C and D are constants, the actual solution depends upon whether $b^2 > \omega^2$, $b^2 = \omega^2$ or $b^2 < \omega^2$.

Case 1: Heavy damping or over damping ($b^2 > \omega^2$)

In this case, $\sqrt{b^2-\omega^2}$ is real and less than b, therefore in equation (4) both the exponents are negative. It means that the displacement 'y' of the particle decreases continuously with time. That is, the particle when once displaced returns to its equilibrium position slowly without performing any oscillations (Fig.8). Such a motion is called ***'overdamped'*** or ***'aperiodic'***motion. This type of motion is shown by a pendulum moving in thick oil or by a deadbeat moving coil galvanometer.

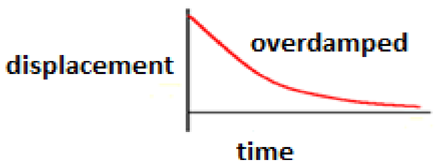


Figure 4: Heavy damping or overdamped motion

Case 2: Critical damping ($b^2 = \omega^2$)

By substituting $b^2 = \omega^2$ in equation (4) the solution does not satisfy equation (2).Hence we consider the case when $\sqrt{b^2-\omega^2}$ is not zero but is a very small quantity β. The equation (4) then can be written as

$$y = C\,e^{(-b+\beta)t} + D\,e^{(-b-\beta)t}$$

$$y = e^{-bt}\,(C\,e^{\beta t} + D\,e^{-\beta t})$$

since β is small, we can approximate,

$e^{\beta t} = 1+\beta t$ *and* $e^{-\beta t} = 1-\beta t$, on the basis of exponential series expansion

$$y = e^{-bt}\left[C(1+\beta t) + D(1-\beta t)\right]$$

$y = e^{-bt}\left[(C+D) + \beta t\,(C-D)\right]$ which is the product of terms

As 't' is present in e^{-bt} and also in the term $\beta t\,(C-D)$, both of them contribute to the variation of y with respect to time. But by the virtue of having t in the exponent, the term e^{-bt} predominantly contributes to the equation. Only for small values of t, the term $\beta t(C-D)$ contributes in a magnitude comparable to that of e^{-bt}. Therefore, though y decreases throughout with increase of t, the decrement is slow in the beginning, and then decreases rapidly to approach the value zero. i.e., the body attains equilibrium position (Fig.5). Such damping motion of a body is called **critical damping**. This type of motion is exhibited by many pointer instruments such as voltmeter, ammeter...etc in which the pointer moves to the correct position and comes to rest without any oscillation.

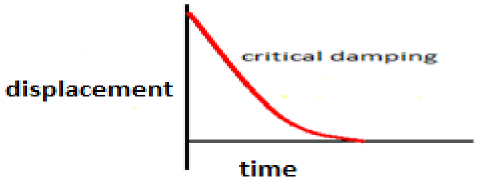


Figure 5: Critical damping motion.

Case 3: Low damping ($b^2 < \omega^2$)

This is the actual case of damped harmonic oscillator. In this case $\sqrt{b^2-\omega^2}$ is imaginary. Let us write

$$\sqrt{b^2-\omega^2} = i\sqrt{\omega^2-b^2} = i\beta'$$

where $\beta' = \sqrt{\omega^2-b^2}$ and $i = \sqrt{-1}$.

Then, equation (4) becomes

$$y = Ce^{(-b+i\beta')t} + De^{(-b-i\beta')t}$$

$$y = e^{-bt}(Ce^{i\beta' t} + De^{-i\beta' t})$$

$$y = e^{-bt}(C(\cos\beta' t + i\sin\beta' t) + D(\cos\beta' t - i\sin\beta' t))$$

$$y = e^{-bt}\left[(C+D)\cos\beta' t + i(C-D)\sin\beta' t)\right]$$

Rewriting $C+D = A\sin\phi$ and $i(C-D) = A\cos\phi$,

Where A and ϕ are constants.

$$y = e^{-bt}\left[A\sin\phi\cos\beta' t + A\cos\phi\sin\beta' t\right]$$

$$y = Ae^{-bt}\sin(\beta' t + \phi)$$

This equation $y = Ae^{-bt}\sin(\beta' t + \phi)$ represents displacement in the **damped harmonic oscillations**. The oscillations are not simple harmonic because the amplitude (Ae^{-bt}) is not constant and decreases with time (t). However, the decay of amplitude depends upon the damping factor b. This motion is known as **under damped motion** (Fig.6). The motion of pendulum in air and the motion of ballistic coil galvanometer are few of the examples of this case.

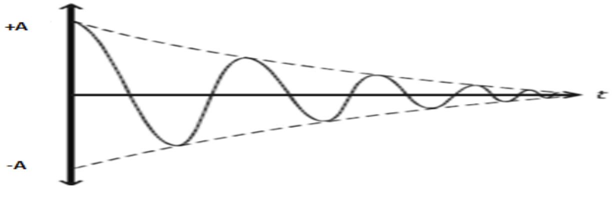


Figure 6: Under damped motion

The time period of damped harmonic oscillator is given by

$$T = \frac{2\pi}{\beta'} = \frac{2\pi}{\sqrt{\omega^2 - b^2}} \frac{2\pi}{\sqrt{\frac{k}{m} - \frac{r^2}{4m^2}}}$$

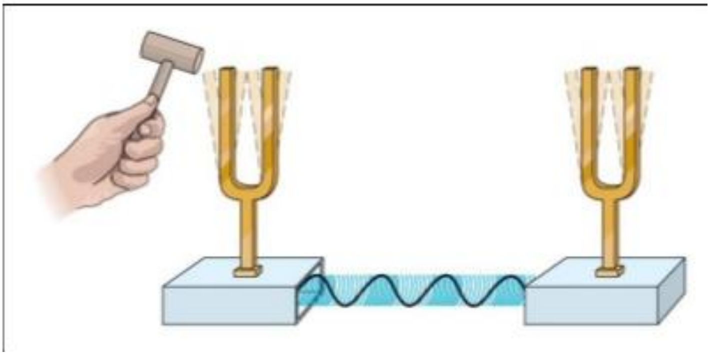
The frequency of damped harmonic oscillator is given by

$$n = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m} - \frac{r^2}{4m^2}}$$

Forced Vibrations:

In the case of damped vibrations, the amplitude of vibrations decreases with the time exponentially due to dissipation of energy and the body eventually comes to rest. When a body experiences vibrations due to the influence of an external driving periodic force, the body can continue its vibration without coming to a rest. Such vibrations are called ***forced vibrations***.

For example: When a tuning fork is struck on a rubber pad and its stem is placed on a table, the table is set in vibrations with the frequency of the fork. These oscillations of the table are the **forced oscillations/vibrations**.



"So forced vibrations can also be defined as the vibrations in which the body vibrates with frequency other than natural frequency of the body, and they are due to applied external periodic force".

DIFFERENTIAL EQUATION OF FORCED VIBRATIONS AND SOLUTION

Suppose a particle of mass m is connected to a spring. When it is displaced and released it starts oscillating about a mean position. The particle is driven by external periodic force $(F_o \sin \omega_d t)$. The oscillations experiences different kinds of forces viz,

1) A restoring force proportional to the displacement but oppositely directed, is given by
   $F_{restoring} = -ky$, where k is known as force constant.

2) A frictional or damping force proportional to velocity but oppositely directed, is given by
   $$F_{damping} = -r\frac{dy}{dt}$$, where r is the frictional force/unit velocity.

3) The applied external periodic force, it is represented by $(F_o \sin\omega_d t)$, where F_o is the maximum value of the force and ω_d is the angular frequency of the driving force.

The total force acting on the particle is given by,

$$F_{net} = F_{external} + F_{restore} + F_{damp} = F_o \sin\omega_d t - r\frac{dy}{dt} - ky$$

By Newton's second law of motion $F = m\frac{d^2y}{dt^2}$.

$$\text{Hence, } m\frac{d^2y}{dt^2} = F_o \sin\omega_d t - r\frac{dy}{dt} - ky$$

The differential equation of forced vibrations is given by

$$\frac{d^2y}{dt^2} + \frac{r}{m}\frac{dy}{dt} + \frac{k}{m}y = \frac{F_o \sin\omega_d t}{m} \text{-----------}(1)$$

Substitute $\frac{r}{m} = 2b$, $\frac{k}{m} = \omega^2$ and $\frac{F_o}{m} = f$,
then equation (1) becomes

$$\frac{d^2y}{dt^2} + 2b\frac{dy}{dt} + \omega^2 y = f\sin\omega_d t \text{-----------}(2)$$

Where b is the damping constant,
F_o is the amplitude of the external driving force
and ω_d is the angular frequency of the external force.

The solution for the differential equation (2) of the forced oscllations will be in the form

$$y = A\sin(\omega_d t - \phi) \text{ ---------- } (3)$$

Where, A is the amplitude of the forced vibrations.

Substituting the value of A in equation (3) we get the solution of the differential equation of the **Forced Harmonic Oscillator** is

$$y = \frac{f}{\sqrt{(\omega^2 - \omega_d^2)^2 + 4b^2\omega_d^2}} \sin(\omega_d t - \phi) \text{ ----------- } (4)$$

The above equation gives the instantaneous displacement y of the oscillating system.

$$\text{Phase} \quad \phi = \tan^{-1}\left[\frac{2b\omega_d}{\omega^2 - \omega_d^2}\right] \text{ ------- } (5)$$

From the above equations it is clear that the amplitude and phase ϕ of the forced oscillations depend upon $(\omega^2 - \omega_d^2)$, i.e., these depend upon the driving frequency (ω_d) and the natural frequency (ω) of the oscillator.

We shall study the behavior of amplitude and phase in three different stages of frequencies i.e, ***low frequency, resonant frequency and high frequency.***

Case I: When driving frequency is low i.e., $(\omega_d << \omega)$.

In this case, amplitude of the vibrations is given by

$$A = \frac{f}{\sqrt{\left(\omega^2 - \omega_d^2\right)^2 + 4b^2\omega_d^2}}$$

As $\omega_d << \omega$, $\sqrt{\left(\omega^2 - \omega_d^2\right)^2} = \sqrt{\left(\omega^2\right)^2} = \omega^2$ and $4b^2\omega_d^2 = 0 \because \omega_d \to 0$

$$A = \frac{f}{\omega^2} = \frac{F_0/m}{k/m} = \frac{F_0}{k} \qquad \because f = \frac{F_0}{m} \text{ and } \omega^2 = \frac{k}{m}$$

$$\text{and} \quad \varphi = \tan^{-1}\left[\frac{2b\omega_d}{(\omega^2 - \omega_d^2)}\right] = \tan^{-1}(0) = 0$$

This shows that the amplitude of the vibration is independent of frequency of the driving force and is dependent on the magnitude of the driving force and the force-constant (k). In such a case the force and displacement are always in phase.

Case II: When $\omega_d = \omega$ i.e., frequency of the driving force is equal to the natural frequency of the body. This frequency is called resonant frequency. In this case, amplitude of vibrations is given by

$$A = \frac{f}{\sqrt{(\omega^2 - \omega_d^2)^2 + 4b^2\omega_d^2}}$$

$$A = \frac{f}{2b\omega_d} = \frac{F_o/m}{(r/m)\omega_d} = \frac{F_o}{r\omega_d}$$

$$\text{and } \phi = \tan^{-1}\left[\frac{2b\omega_d}{\omega^2 - \omega_d^2}\right]$$

$$= \tan^{-1}\left[\frac{2b\omega_d}{0}\right] = \tan^{-1}(\infty) = \frac{\pi}{2}$$

Under this situation, the amplitude of the vibrations becomes maximum and is inversely proportional to the damping coefficient. For small damping, the amplitude is large and for large damping, the amplitude is small. The displacement lags behind the force by a phase π/2.

Case III: When $(\omega_d >> \omega)$ i.e., the frequency of force is greater than the natural frequency of the body. In this case, amplitude of the vibrations is given by

$$A = \frac{f}{\sqrt{(\omega^2 - \omega_d^2)^2 + 4b^2\omega_d^2}}$$

$$\omega_d >> \omega, \therefore (\omega^2 - \omega_d^2)^2 = \omega_d^4 \text{ since } \omega_d \text{ is very large}, \omega_d^4 >>> 4b^2\omega_d^2$$

$$\therefore A = \frac{f}{\omega_d^2} = \frac{F_0/m}{\omega_d^2} = \frac{F_0}{m\omega_d^2}$$

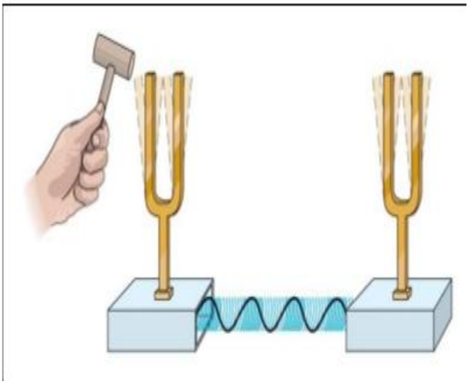
$$\text{and } \varphi = \tan^{-1}\left[\frac{2b\omega_d}{(\omega^2 - \omega_d^2)}\right]$$

$$= \tan^{-1}\left[-\frac{2b}{\omega_d}\right] = \tan^{-1}(-0) = \pi$$

This shows that amplitude depends on the mass and continuously decreases as the driving frequency ω_d is increased and phase difference towards π.

Resonance:

If we bring a vibrating tuning fork near another stationary tuning fork of the same natural frequency as that of vibrating tuning fork, we find that both the tuning forks start vibrating with the same frequency and the amplitude will be maximum. This phenomenon is known as Resonance.



Resonance is a phenomenon in which a body vibrates with its natural frequency with maximum amplitude under the influence of an external vibration with the same frequency.

Theory of resonant vibrations:

(a) Condition of amplitude resonance. In case of forced vibrations, the expression for amplitude A and phase ϕ is given by,

$$A=\frac{f}{\sqrt{(\omega^2-\omega_d^2)^2+4b^2\omega_d^2}} \text{ and } \phi=\tan^{-1}\left[\frac{2b\omega_d}{\omega^2-\omega_d^2}\right]$$

The amplitude expression shows the variation with the frequency of the driving force ω_d. For a particular value of ω_d, the amplitude becomes maximum. The phenomenon of amplitude becoming a maximum is known as amplitude resonance. The amplitude is maximum when $\sqrt{(\omega^2-\omega_d^2)^2+4b^2\omega_d^2}$ is minimum. If the damping is small i.e., b is small, the condition of maximum amplitude reduced to $A_{max}=\frac{f}{2b\omega_d}$.

(b) Sharpness of the resonance.

We have seen that the amplitude of the forced oscillations is maximum when the frequency of the applied force is at resonant frequency. If the frequency changes from this value, the amplitude falls. When the fall in amplitude for a small change from the resonance condition is very large, the resonance is said to be sharp and if the fall in amplitude is small, the resonance is termed as flat. Thus the term sharpness of resonance can be defined as the rate of fall in amplitude, with respect to the change in forcing frequency on either side of the resonant frequency.

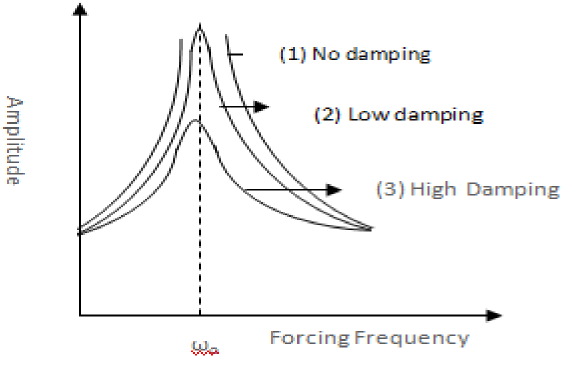


Figure 7: Effect of damping on sharpness of resonance

Figure 7, shows the variation of amplitude with forcing frequency for different amounts of damping. Curve (1) shows the variation of amplitude when there is no damping i.e., b=0. In this case the amplitude is infinite at $\omega_d = \omega$. This case is never realized in practice due to friction/dissipation forces, as a slight damping factor is always present. Curves (2) and (3) show the variation of amplitude with respect to low and high damping. It can be seen that the resonant peak moves towards the left as the damping factor is increased. It is also observed that the value of amplitude, which is different for different values of b (damping), diminishes as the value of b increases. This indicates that the smaller is the damping, sharper is the resonance or large is the damping, flatter is the resonance.

SI No	Sample Questions	CO
1	What is simple harmonic motion?	1
2	Write the general equation representing SHM.	1
3	List any two characteristics of SHM.	1
4	A particle executes a S.H.M. of period 10 seconds and amplitude of 2 meter. Calculate its maximum velocity.	3
5	Hydrogen atom has a mass of 1.68×10^{-27} kg, when attached to a certain massive molecule it oscillates as a classical oscillator with a frequency10^{14} cycles per second and with amplitude of 10^{-10} m. Calculate the acceleration of the oscillator.	3
6	A body executes S.H.M such that its velocity at the mean position is 4cm/s and its amplitude is 2cm. Calculate its angular velocity.	3
7	What is free vibration?	1
8	What is damped vibration?	1
9	Give any two examples for damped vibration.	1
10	What is forced vibration?	1
11	Given two vibrating bodies what is the condition for obtaining resonance?	2
12	Explain why a loaded bus is more comfortable than an empty bus?	2
13	What is a simple harmonic oscillator?	1
14	What is torsional oscillation?	1
15	Displacement of a particle of mass 10g executes SHM given by $x = 15 sin\left(\frac{2\pi t}{T} + \varphi\right)$ and its displacement at t=0 is 3cm where the amplitude isn15cm. Calculate the initial phase of the particle.	3
16	Name the two forces acting on a system executing damped vibration.	2
17	How critical damping is beneficiary in automobiles?	1
18	What is restoring force?	1
19	Every SHM is periodic motion but every periodic motion need	2

	not be SHM. Why? Support your answer with an example.	
20	Distinguish between linear and angular harmonic oscillator?	1
21	Setup the differential equation for SHM.	2
22	Define the terms (i) time period (ii) frequency (iii) phase and (iii) angular frequency of oscillations	1
23	What is the phase difference between (i) velocity and acceleration (ii) acceleration and displacement of a particle executing SHM?	2
24	Show graphically the variation of displacement, velocity and acceleration of a particle executing SHM.	1
25	Explain the oscillations of a mass attached to a horizontal spring. Hence deduce an expression for its time period.	2
26	Derive an expression for the time period of a body when it executes angular SHM	1
27	What is damping? On what factors the damping depends?	1
28	What are damped vibrations? Establish the differential equation of motion for a damped harmonic oscillator and obtain an expression for displacement. Discuss the case of heavy damping, critical damping and low damping.	2
29	What do you mean by forced harmonic vibrations? Discuss the vibrations of a system executing simple harmonic motion when subjected to an external force.	2
30	What is driven harmonic oscillator? How does it differ from simple and damped harmonic oscillator?	2
31	What is resonance? Explain the sharpness of resonance.	2
32	Illustrate an example to show that resonance is disastrous sometimes.	2

SI No	**Numerical Problems**	**CO**
1.	A particle executes SHM of period 31.4 second and amplitude 5cm. Calculate its maximum velocity and maximum acceleration. **Solution:** The maximum velocity at y =0 in $v = \omega\sqrt{A^2 - y^2}$, $\therefore v_{max} = \omega A$ $\omega = \frac{2\pi}{T} = \frac{2\pi}{31.4} = 0.2 radian$ $\therefore v_{max} = 0.2 \times 5 = 1.0 cm/\sec = 0.01 m/s$ At the maximum displacement, i.e., at the extreme position x=A, maximum acceleration is - $\omega^2 A$ $a = -0.2^2 \times 5 = -0.002 m/s^2$	3
2.	A circular plate of mass 4kg and diameter 0.10 metre is suspended by a wire which passes through its centre. Find the period of angular oscillations for small displacement if the torque required per unit twist of the wire is 4 x 10^{-3} N-m/radian. **Solution:** **Moment of Inertia** $I = \frac{MR^2}{2} = \frac{4(0.05)^2}{2} = 0.005 kgm^2$ **Time period is given by** $T = 2\pi\sqrt{\frac{I}{C}} == 2 \times 3.14 \times \sqrt{\frac{0.005}{4x10^{-3}}}$ **=7.021s.**	3
3.	A mass of 6kg stretches a spring 0.3m from its equilibrium position. The mass is removed and another body of mass 1kg is hanged from the same spring. What would be the period of motion if the spring is now stretched and released? **Solution:** $F=ky,\ k=\frac{F}{y}=\frac{mg}{y}=\frac{6\times9.8}{0.3}=196 N/m$ and $T=2\pi\sqrt{\frac{m}{k}}=2\times3.14\sqrt{\frac{1}{196}}=0.45s$	3

4.	A vibrating system of natural frequency 500 cycles/sec, is forced to vibrate with a periodic force/unit mass of amplitude 100 x 10^{-5} N/kg in the presence of a damping/ unit mass of 0.01 x 10^{-3} rad/s. Calculate the maximum amplitude of vibration of the system. **Given** :Natural frequency = 500 cycles/sec, Amplitude of the force / unit mass , Fo/m=100 x 10^{-5} N/kg, Damping coefficient , r/m = 0.01 x 10^{-3} rad/s **Solution:** Maximum amplitude of vibration $A=\frac{f}{2b\omega}=\frac{F_o/m}{(r/m)\omega}$ $\therefore A=\frac{100\times10^{-5}}{0.01\times10^{-3}\times2\times\pi\times500}=0.0318m.$ $\therefore$ The maximum amplitude of vibration of the system is 0.0318meter	3
5.	A circuit has an inductance of $1/\pi$ henry and resistance 100Ω. An A. C. supply of 50 cycles is applied to it. Calculate the reactance and impedance offered by the circuit. **Solution:** The inductive reactance is $X_L=\omega_d L=2\pi nL$ Here $\omega_d=2\pi n=2\pi$ x 50= 100π rad/sec and L=$1/\pi$ Henry. $X_L=\omega_d L=2\pi nL=2\pi$ x 50x $1/\pi$ =100Ω. The impedance is $Z=\sqrt{R^2+X_L^2}=\sqrt{100^2+100^2}=141.4\Omega$	3
6.	A series LCR circuit has L=1mH, C=0.1μF and R=10Ω.Calculate the resonant frequency of the circuit. **Solution:** The resonant angular frequency of the circuit is given by $\omega=\frac{1}{\sqrt{LC}}$ Here L=1mH and C=0.1μF $\omega=\frac{1}{\sqrt{10^{-3}x10^{-7}}}=10^5 rad/s$	3

Reference Books	
1	Oscillations and Waves, Satya Prakash, Pragati Prakshan, third edition, 2005.
2	Engineering Physics Hitendra K Malik and A K Singh, Tata McGraw Hill Education Private Limited, 2009, ISBN: 978-0-07-067153-9.
3	A Text book of Engineering Physics Dr. M N Avadhanulu, Dr. P. G. Kshirsagar, S. Chand & Company Private limited. Revised edition 2015.
4	Engineering Physics R K Gaur and S L Gupta, Dhanpat Rai Publications, Revised edition 2011.

Appendix:

Example for Electrical Resonance: LCR circuit

An L-C-R circuit fed by an alternating emf is a classic example for a forced harmonic oscillator. Consider an electric circuit containing an inductance L, capacitance C and resistance R in series as shown in the figure 8. An alternating emf has been applied to a circuit is represented by $E_o \sin \omega_d t$.

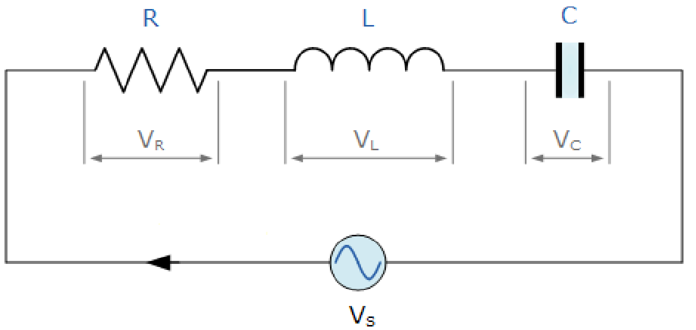


Figure 8: Series LCR circuit.

Let q be the charge on the capacitor at any instant and I be the current in the circuit at any instant. The potential difference across the capacitor is $\frac{q}{C}$, the back emf due to self inductance in the inductor is $L\frac{dI}{dt}$ and, the potential drop across the resistor is IR. The sum of voltages across the three LCR elements illustrated must be equal the voltage supplied by the source element. Hence the voltage equation at any instant is given by,

$$V_L + V_R + V_C = V_S(t)$$

$$L\frac{dI}{dt} + IR + \frac{q}{C} = E_o \sin\omega_d t$$

Differentiating, we get

$$L\frac{d^2I}{dt^2}+R\frac{dI}{dt}+\frac{1}{C}\frac{dq}{dt}=\omega_d E_o \cos\omega_d t$$

$$\text{but}\ \frac{dq}{dt}=I,\ \therefore \qquad L\frac{d^2I}{dt^2}+R\frac{dI}{dt}+\frac{I}{C}=\omega_d E_o \cos\omega_d t$$

$$\frac{d^2I}{dt^2}+\left(\frac{R}{L}\right)\frac{dI}{dt}+\left(\frac{1}{LC}\right)I=\left(\frac{\omega_d E_o}{L}\right)\cos\omega_d t$$

$$\frac{d^2I}{dt^2}+\left(\frac{R}{L}\right)\frac{dI}{dt}+\left(\frac{1}{LC}\right)I=\left(\frac{\omega_d E_o}{L}\right)\sin\left(\omega_d t+\frac{\pi}{2}\right)-----(1)$$

This is the differential equation of the ***forced oscillations*** in the electrical circuit. It is similar to equation of motion of a mechanical oscillator driven by an external force.

$$\frac{d^2y}{dt^2}+2b\frac{dy}{dt}+\omega^2 y=\frac{F_o}{m}\sin\omega_d t\ \text{-------------------------}(2)$$

The explicit and precise connection with the mechanical oscillation equation is given below:

Displacement:	y	⟷	I
Velocity :	$\frac{dy}{dt}$	⟷	$\frac{dI}{dt}$
Damping constant:	$2b$	⟷	R/L
Natural frequency:	ω	⟷	$1/\sqrt{LC}$
External force :	$F(t)$	⟷	E_o
And	F_o/m	⟷	$\omega_d E_o/m$

Forced oscillations (refer previous section)	LCR Circuit application
Starting with equation from forced vibrations, we have $$\frac{d^2y}{dt^2} + 2b\frac{dy}{dt} + \omega^2 y = \frac{F_o}{m}\sin\omega_d t$$ Amplitude of mechanical vibrations is given by $$A = \frac{F_o/m}{\sqrt{(\omega^2 - \omega_d^2)^2 + 4b^2\omega_d^2}}$$	The equivalent equation for LCR circuit is given by $$\frac{d^2I}{dt^2} + \left(\frac{R}{L}\right)\frac{dI}{dt} + \left(\frac{1}{LC}\right)I = \left(\frac{\omega_d E_o}{L}\right)\sin\left(\omega_d t + \frac{\pi}{2}\right)$$ Amplitude of current I_o is given by $$I_O = \frac{\frac{\omega_d}{L}E_o}{\sqrt{(\omega^2 - \omega_d^2)^2 + 4b^2\omega_d^2}}$$ Substitute for $\omega^2 = \frac{1}{LC}$ and $4b^2 = \frac{R^2}{L^2}$, we get $$I_o = \frac{\frac{\omega_d}{L}E_o}{\sqrt{\left(\frac{1}{LC} - \omega_d^2\right)^2 + \frac{R^2}{L^2}\omega_d^2}}$$ In the denominator, multiply and divde the term $\frac{1}{LC}$ by ω_d and ω_d^2 by L, we get $$I_o = \frac{\frac{\omega_d}{L}E_o}{\sqrt{\left[\left(\frac{1}{LC}\frac{\omega_d}{\omega_d}\right) - \left(\frac{L}{L}\omega_d^2\right)\right]^2 + \frac{R^2}{L^2}\omega_d^2}}$$ $$I_o = \frac{\frac{\omega_d}{L}E_o}{\sqrt{\frac{\omega_d^2}{L^2}\left[\left(\frac{1}{C\omega_d}\right) - (L\omega_d)\right]^2 + \frac{R^2}{L^2}\omega_d^2}}$$

	$$I_o = \frac{\frac{\omega_d}{L} E_o}{\frac{\omega_d}{L}\sqrt{\left(\frac{1}{C\omega_d} - L\omega_d\right)^2 + R^2}}$$ substitute $\frac{1}{C\omega_d} = X_C$ (Capacitive reactance) and $L\omega_d = X_L$ (Inductive reactance) $$I_o = \frac{E_o}{\sqrt{(X_C - X_L)^2 + R^2}} = \frac{E_o}{\sqrt{[(X_L - X_C)]^2 + R^2}}$$

The solution of the equation (1) for the current at any instant in the circuit is of the form

$$I = \frac{E_o}{\sqrt{R^2 + \left(\omega_d L - \frac{1}{\omega_d C}\right)^2}} \sin(\omega_d t - \phi) = I_o \sin(\omega_d t - \phi) \text{-------}(3)$$

Electrical Impedance: The ratio of amplitudes of alternating emf and current in ac circuit is called the electrical impedance of the circuit. *i.e.*, $Z = \frac{E_o}{I_o} \quad \therefore I_o = \frac{E_o}{Z}$

The amplitude of the current is $I_o = \frac{E_o}{\sqrt{R^2 + \left(\omega_d L - \frac{1}{\omega_d C}\right)^2}} = \frac{E_o}{Z}$

Impedance of the circuit $Z = \sqrt{R^2 + \left(\omega_d L - \frac{1}{\omega_d C}\right)^2}$ ----------(4).

The quantity $\left(\omega_d L - \frac{1}{\omega_d C}\right)$ is the net reactance of the circuit which is the difference between the inductive reactance $(X_L) = \omega_d L$ and the capacitive reactance $(X_C) = \frac{1}{\omega_d C}$. Equation (3) shows that the current I lags in phase with the applied emf $E_o \sin\omega_d t$ by an angle ϕ and is given by

$$\phi = \tan^{-1}\left(\frac{\omega_d L - \frac{1}{\omega_d C}}{R}\right) = \tan^{-1}\left(\frac{X_L - X_C}{R}\right)$$

The following three cases arise:

1. When $X_L > X_C$, ϕ is positive, that is, the current lags behind the emf. Circuit is inductive.
2. When $X_L < X_C$, ϕ is negative, that is, the current leads behind the emf. Circuit is capacitive.
3. When $X_L = X_C$, ϕ is zero, that is, the current is in phase with the emf. Circuit is resistive.

Electrical Resonance: According to equation (4), the current have its maximum amplitude when

$$X_L - X_C = 0 \;\; or \;\; \omega_d L - \frac{1}{\omega_d C} = 0 \;\; or \;\; \omega_d L = \frac{1}{\omega_d C} \;\; or \;\; \omega_d = \frac{1}{\sqrt{LC}}$$

Where ω_d is the angular frequency of the applied emf, while $\omega = \frac{1}{\sqrt{LC}}$ is the (angular) natural frequency of the circuit.

Hence, the maximum amplitude of the current oscillations occurs when the frequency of the applied emf is exactly equal to the natural (un-damped) frequency of the electrical circuit. This is the condition of **electrical resonance**.

Sharpness of resonance and Bandwidth:

When an alternating emf is applied to an LCR circuit, electrical oscillations occur in the circuit with the frequency $\omega_d$is equal to the applied emf. The amplitude of these oscillations (current amplitude) in the circuit is given by

$$I_o = \frac{E_o}{\sqrt{R^2 + \left(\omega_d L - \frac{1}{\omega_d C}\right)^2}} = \frac{E_o}{Z}, \text{ Where } Z \text{ is the impedance of the circuit.}$$

At resonance, when the frequency ω_d of the applied emf is equal to the natural frequency $\omega = \frac{1}{\sqrt{LC}}$ of the circuit, the current amplitude I_o is maximum and is equal to E_o/R. Thus at resonance the impedance Z of the circuit is R. At other values of ω_d, the current amplitude I_o is smaller and the impedance Z is larger than R.

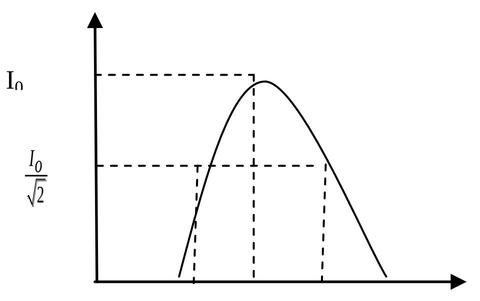


Figure 9: Graphical] **uency.**

The variation of the current amplitude I_o with respect to applied emf frequency ω_d is shown in the figure 9. I_o attains maximum value (E_o/R) when ω_d has resonant value ω and decreases as ω_d changes from ω. The rapidity with which the current falls from its resonant value (E_o/R) with change in applied frequency is known as the **sharpness of resonance.** It is measured by the ratio of the resonant frequency $\omega$to the difference of two frequencies ω_1 and $\omega_2$taken at $\frac{1}{\sqrt{2}}$ of the resonant (ω) value.

i.e**.,** **Sharpness of resonance (Q)** $= \frac{\omega}{\omega_2 - \omega_1}$,

ω_1 and ω_2 are known as the **half power frequencies**.

The difference of half power frequencies, ω_1 - ω_2 is known as **"band-width"**. The smaller is the bandwidth, the sharper is the resonance.