Unit-I-Elasticity and Oscillations

Introduction:

Materials are classified as Rigid, Elastic and Plastic based on the behavior of the material on the application of external force. The applied force is called deforming force.

A rigid body is a body which does not undergo any deformation when external forces act on it. When forces are applied on a rigid body the distance between any two particles of the body will remain unchanged however large the force may be. In actual practice no body is perfectly rigid. For practical purposes solid bodies are taken as rigid bodies.

When a body is acted upon by a suitable force, it undergoes a change in form, this change in form is called **Deformation**. The change could be either in shape or size or even both. If the body recovers its original state on the removal of deforming force, then the body is called an **Elastic body**.eg. Quartz

"Elasticity is the property of the material of a body by virtue of which it regains its original shape and size after the deforming forces are removed".

If the body does not show any tendency of returning back to its original or initial state and stays in the changed form after the withdrawal of external force, then it is said to be in Plastic state. eg Clay.

"Plasticity is the property of the material of a body by virtue of which it fails to regain its original shape and size after the deforming forces are removed".

What happens to the body when these forces are removed?

The external forces acting on the body compel the molecules to change their position. Due to these relative molecular displacements, internal forces are developed within the body that tend to oppose the external deforming force. When the deforming forces are removed, the internal forces will tend to regain the shape and size of the body.

How does one account for this?

Under the action of the external force, the body changes its form because; the molecules inside it are displaced from their previous positions. While they are displaced, the molecules develop a tendency to come back to their original positions, because of intermolecular binding forces. The aggregate of the restoration tendency exhibited by all the molecules of the body manifests as a balancing force or restoring force counteracting the external force.

Important terms to be understood

Deforming Force:The applied external force which causes deformation is called deforming force.

Restoring force: This reactional force which is developed in a deformed body is called restoring force.

LOAD: It is the external forces acting on a body is the load and its effect is to change the form or the dimensions of the body.

STRESS: Stress is the ratio of Restoring force to the area of its application.

Unit of stress is Nm^{-2}.

Normal stress: Restoring force per unit area perpendicular to the surface is called normal stress.

Tangential stress: Restoring force parallel to the surface per unit area is calledtangential stress.

STRAIN:
The deformation produced by the external force accompanying a change in dimensions or form of the body is called strain. It is the **ratio of change in dimensions of the body to its original dimensions.**
The way in which the change in dimensions is produced depends upon the form of the body and the manner in which the force is applied.

Deformation is of three types, resulting in three types of strains, defined as follows:

i) **Linear strain or Tensile strain**: If the shape of the body could be approximated to the form of a long wire and if a force is applied at one end along its length keeping the other end fixed, the wire undergoes a change in length.

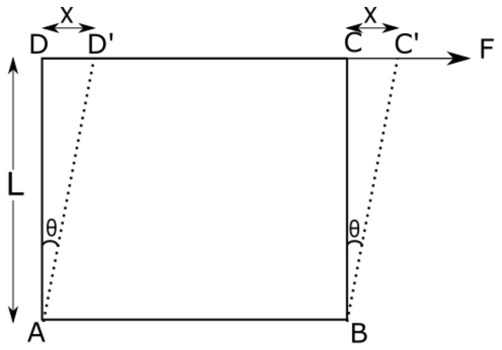
If x is the change in length produced for an original length L then,

$$\textbf{Linear strain} = \frac{change\, in\, length}{original\, length} = \frac{x}{L}$$

ii) **Volume strain Bulk strain :** If a uniform force is applied all over the surface of a body, the body undergoes a change in its volume (however the shape is retained in case of solid bodies). If v is the change in volume to an original volume V of the body, then,

$$\textbf{Volume strain} = \frac{\text{change in volume}}{\text{original volume}} = \frac{v}{V}$$

iii) **Shear strain:** If a force is applied tangentially to a free surface of a body with the other surface is being fixed, its layers slide one over the other; the body experiences a turning effect and changes its shape. This is called **shearing** and the angle through which the turning takes place is called **shearing angle (θ).**



Shearing strain (θ)=$\frac{x}{L}$

HOOKE'S LAW: This is a relation between stress and strain
It states that **"stress is directly proportional to strain" (provided strain is small),**
i.e., stress α strain,

Or, $\frac{stress}{strain} = \text{a constant (E)}$
The constant of proportionality E is called the modulus of elasticity or coefficient of elasticity. (This is the measure of elasticity of the material of the body)

Corresponding to the three types of strain, we have three types of elasticity:

a) Linear Elasticity or Young's modulus, corresponding to linear or tensile strain:
When the deforming force is applied to the body, along a particular direction, the change per unit length in that direction is called longitudinal or linear or elongation strain, and the force developed applied per unit area of cross section is called longitudinal or linear stress. **The ratio of longitudinal stress to linear strain within elastic limit is called the Young's modulus and is denoted by Y.**
If F is the force applied normally, to a cross-sectional area a, then the stress is F/a. If L is original length and x is change in length due to the applied force, the strain is given by x/L,
so that,

$$\mathbf{Y} = \frac{\text{Normal stress}}{\text{Longitudinal strain}} = \frac{F/a}{x/L} = \frac{FL}{ax} \ \mathbf{N/m^2}$$

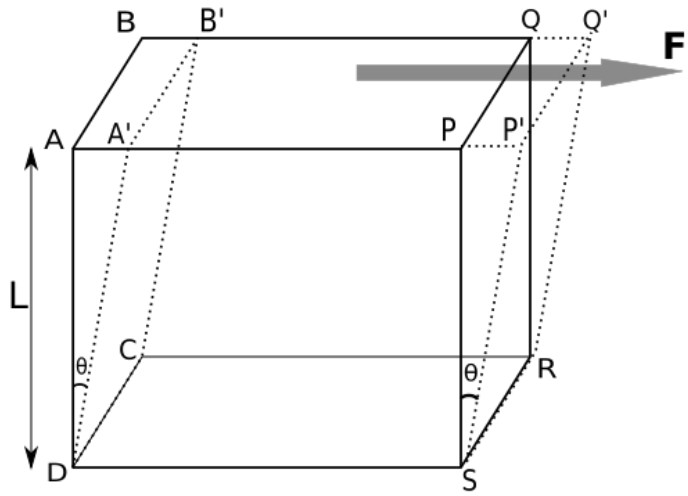
b) Bulk modulus:
When the deforming force is applied uniformly along the normal to the surface of a body, it produces a volume strain (without changing its shape in case of solid bodies). The applied force per unit area gives the normal stress or pressure. **The ratio of normal stress to the volume strain.**
If F is the force applied uniformly and normally on a surface area a, the stress or pressure is F/a or P and if v is the change in volume produced in an original volume V, the strain is given by v/V and therefore

$$\mathbf{K} = \frac{\text{Normal stress}}{\text{Volume strain}} = \frac{F/a}{v/V} = \frac{FV}{av} = \frac{PV}{v} \ \mathbf{N/m^2}$$

NOTE:Bulk modulus is referred to as incompressibility and hence its reciprocal is called compressibility (strain per unit stress).

c) Modulus of Rigidity (corresponding to shear strain) :

In this case, while there is a change in the shape of the body, there is no change in its volume. It takes place by the movement of layers of the body, one over the other.
Consider a rectangular solid cube whose lower face DCRS is fixed, and to whose upper face a tangential force F is applied in the direction as shown. Under the action of this force, the layers of the cube which are parallel to the applied force slide one over the other such that point A shifts to A^1, B to B^1, P to P^1 and Q to Q^1. The body has turned through an angle θ. This angle θ is called the angle of shear or **shearing strain**. Tangential stress is equal to the force F divided by area a of the face APQB.



Hence **tangential stress** = $\frac{F}{a}$

Shearing strain (θ) = PP^1 / PS = $\frac{x}{L}$

The rigidity modulus is defined as the ratio of the tangential stress to the shearing strain.

Rigidity modulus η = $\frac{\text{tangential stress}}{\text{shearing strain.}} = \frac{F/a}{\theta} = \frac{F/a}{x/L} = \frac{FL}{ax}$ **N/m²**

Poisson's Ratio (σ):

In case of any deformation taking place along the length of a body like a wire, due to a deforming force, there is always some change in the thickness of the body. This change which occurs in a direction perpendicular to the direction along which the deforming force is acting is called **lateral change**.

Within elastic limits of a body, the ratio of lateral strain to the longitudinal strain is a constant and is called Poisson's ratio.

If a deforming force acting on a wire of length L produces a change in length x accompanied by a change in diameter of d in it which has a original diameter of D, then lateral strain $\beta = \frac{d}{D}$

and Longitudinal strain $\alpha = \frac{x}{L}$,

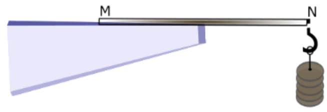
$\therefore$ **Poisson's ratio, σ =** $\frac{\beta}{\alpha} = \frac{Ld}{xD}$

There are no units for Poisson's ratio. It is a dimensionless quantity.
Limits of σ

Theoretically the limiting values of σ are -1 and 0.5. A negative value of σ means that a body on being extended should also expand laterally and one can hardly expect this to happen(we know of no substance so far). Hence the value of **σ is between 0 and 0.5**
In actual practice, the value of σ is found to lie between 0.2 and 0.4.

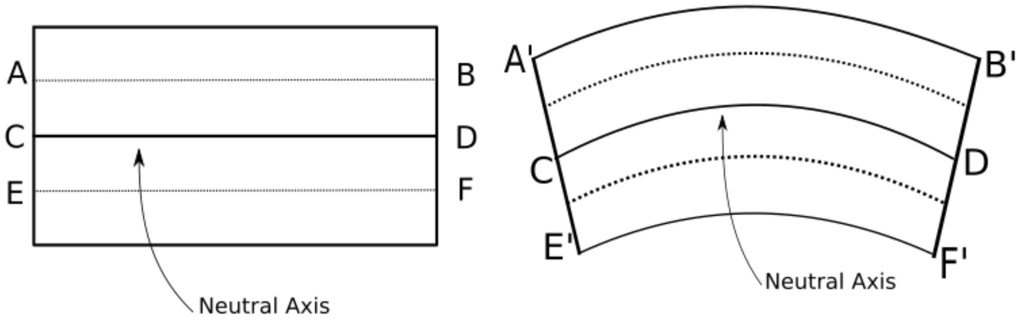
<u>Bending of beams:</u>

A homogenous body of uniform cross section whose length is large compared to its other dimensions is called a beam.



Neutral surface and neutral axis:
Consider a uniform beam MN whose one end isfixed at M. The beam can be thought of consisting of a number of parallel layers and each layer in turn can be thought of as made up of a number of infinitesimally thin straight parallel longitudinal filaments or fibers arranged one closely next to the other in the plane of the layer. If the cross section of the beam along its length and perpendicular to these layers is considered, the filaments of different layers appear like straight lines piled one above the. For a given layer, all its constituent filaments are assumed to undergo identical changes when that layer is strained.

If a load is attached to the free end of the beam, the beam bends. The successive layers along with constituent filaments are strained. A filament like AB of an upper layer will be elongated to A^1B^1 and the one like EF of a lower layer will be contracted to E^1F^1. But there will always be a particular layer whose filaments do not change their length as shown for CD. Such a layer is called neutral surface and the line along which a filament of it is situated is called neutral axis.
The filaments of a neutral surface could be taken as line along which the surface is intercepted by a cross section of the beam considered in the plane of bending as shown.



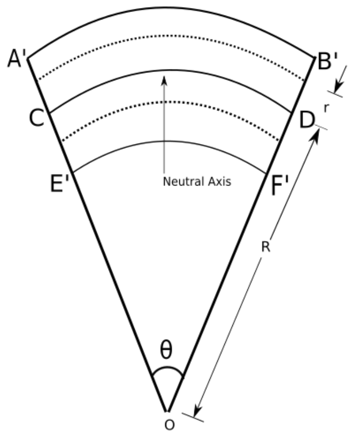
Neutral Surface: It is that layer of a uniform beam which does not undergo any change in its dimensions, when the beam is subjected to bending within its elastic limit.

Neutral axis: It is a longitudinal line along which neutral surface is intercepted by any longitudinal plane considered in the plane of bending.

When a uniform beam is bent, all its layers which are above the neutral surface undergo elongation whereas those below the neutral surface are subjected to compression. As a result, the forces of reaction are called into play in the body of the beam which develop an inward pull towards the fixed end for all the layers above the neutral surface and an outward push directed away from the fixed end for all layer below the neutral surface. These two groups of forces result in a restoring couple which balances the applied couple acting on the beam. The moment of restoring forcesabout the neutral axis is called bending moment. When the beam is in equilibrium, the bending moment and the restoring moments are equal.

Bending moment of a beam:

Consider a uniform beam whose one end is fixed. If now a load is attached to the beam, the beam bends. The successive layers are now strained. A layer like AB which is above the neutral surface will be elongated to A^1B^1 and the one like EF below neutral surface will be contracted to E^1F^1. CD is neutral surface which does not change its length.



The layers of the bent beam can be imagined to form part of concentric circles of varying radii. Let R be the radius of the circle to which the neutral surface forms a part.

$\therefore CD = R\theta$

where θ is the common angle subtended by the layers at common center O of the circles. The layer AB has been elongated to A^1B^1.

$\therefore$ change in length $=A^1B^1-AB$
But $AB=CD=R\theta$
If the successive layers are separated by a distance r then,
$A^1B^1=(R+r)\theta$
$\therefore$ Change in length$=(R+r)\theta-R\theta = r\theta$
But original length = $AB=R\theta$
$\therefore$ Linear strain $= \frac{r\theta}{R\theta} = \frac{r}{R}$
Young's Modulus Y= Longitudinal stress/linear strain
Longitudinal stress = Yx Linear strain

$$= Y\text{x}\frac{r}{R}$$

But stress $= \frac{F}{a}$

Where 'F' is the force acting on the beam and 'a' is the area of the layer AB.

$$\frac{F}{a} = \frac{Yr}{R}$$
$$F = \frac{Yar}{R}$$

Moment of this force about the neutral axis=F x its distance from neutral axis.

$$= F \text{ x } r = Yar^2/R$$

Moment of forces acting on the entire beam = Bending moment $=\Sigma \frac{Yar^2}{R}$

Bending moment $= \frac{Y}{R}\sum ar^2$

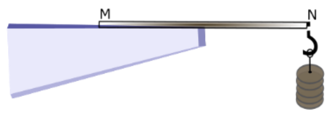
The moment of inertia of a body about a given axis is given by Σmr^2, whereas Σar^2 is called the geometric moment of Inertia Ig.

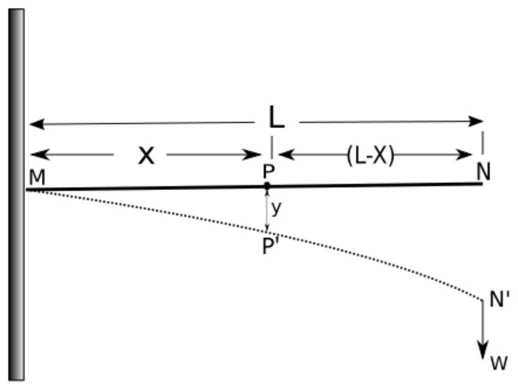
$Ig=\Sigma ar^2 = Ak^2$, where 'A' is the area of cross section of the beam and 'k' is the radius of gyration about the neutral axis.

Bending moment $= \frac{Y}{R} Ig$

Bending moment $= \frac{Y}{R} Ak^2$

Single Cantilever:





If one end of beam is fixed to a rigid support and its other end loaded, then the arrangement is called single cantilever or cantilever.

Consider a uniform beam of length ‘L’ fixed at M. Let a load ‘W’ act on the beam at N.

Consider a point on the free beam at a distance ‘x’ from the fixed end which will be at a distance (L-x) from N. Let P' be its position after the beam is bent.

$\therefore$ Bending moment = Force x Perpendicular distance.

= W(L-x)

But bending moment of a beam is given by $\frac{Y}{R} Ig$

Where R is the radius of curvature of the bent beam, Y is the young’s modulus Ig is the geometric moment of inertia

$$\frac{Y}{R} Ig = W(L-x) \text{ ----------------(1)}$$

$$\frac{1}{R} = \frac{W(L-x)}{YIg} \text{ -------------------(2)}$$ (1/R is the curvature of the neutral axis)

But if y is the depression of the point P, then it can be shown that

$$\frac{1}{R} = \frac{d^2y}{dx^2} \text{ ---------------------(3)}$$

Comparing equations (2) and (3)

$$\frac{d^2y}{dx^2} = \frac{W(L-x)}{YIg}$$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{w(L-x)}{YI_g}$$

Integrating both sides

$$\frac{dy}{dx} = \frac{w}{YI_g}\left[Lx - \frac{x^2}{2}\right] + C_1 \text{ ---------------(4)}$$

C is constant of integration

But dy/dx is the slope of the tangent drawn to the bent beam at a distance x from the fixed end. When x=0, it refers to the tangent drawn at M, where it is horizontal. Hence (dy/dx)=0 at x=0. Introducing this condition in equation (4) we get $0=C_1$

Equation 4 becomes

$$\frac{dy}{dx} = \frac{W}{YIg}\left[Lx - \frac{x^2}{2}\right]$$

$$dy = \frac{W}{YIg}\left[Lx - \frac{x^2}{2}\right]dx$$

Integrating both sides we get

$$y = \frac{W}{YIg}\left[\frac{Lx^2}{2} - \frac{x^3}{6}\right] + C_2 \text{ ----------------- (5)}$$

where C_2 is constant of integration, y is the depression produced at known distance from the fixed end.Therefore, when x=0, it refers to the depression at M, where there is obviously no depression. Hence y=0 at x=0. Introducing this condition in equation 5 we get

$$y = \frac{W}{YIg}\left[\frac{Lx^2}{2} - \frac{x^3}{6}\right]$$

At the loaded end, $y=y_0$ and x=L

Therefore $$y_0 = \frac{W}{YIg}\left[\frac{L^3}{2} - \frac{L^3}{6}\right]$$

Depression produced at loaded end is

$$y_0 = \frac{WL^3}{3YIg}$$

Therefore, the young's modulus of the material of the cantilever is

$$Y = \frac{WL^3}{3y_0 Ig} \text{ -----------------(6)}$$

Case (a):

If the beam is having rectangular cross-section, with breadth b and thickness d then,

$$Ig = \frac{bd^3}{12} \text{ -----------------(7)}$$

Substituting equation (7) in equation (6) we get

$$Y = \frac{WL^3}{3y_0} x \frac{12}{bd^3}$$

$$Y = \frac{4WL^3}{Y_0 bd^3}$$

Case(b):

If the beam is having a circular cross section of radius r then,

$$Ig = \frac{\Pi r^4}{4} \text{ ---------------------(8)}$$

Substituting equation (8) in equation (7) we get

$$Y = \frac{4WL^3}{3\Pi y_0 r^4}$$

Torsion of a cylinder

A long body which is twisted around its length as an axis is said to be under torsion. The twisting is brought into effect by fixing one end of the body to a rigid support and applying a suitable couple at the other end. The elasticity of a solid, long uniform cylindrical body under torsion can be studied, by imagining it to be consisting of large number of coaxial, hollow cylindersone within the other. The applied twisting couple is calculated in terms of the rigidity modulus of the body.

Expression for twisting couple or Torsion of a cylindrical rod:

Consider a long cylindrical rod of length 'L' and radius 'R'.Its upper end is clamped and the rod is twisted through an angle θ, by applying a couple to its lower end in a plane perpendicular to its length.

Due to the property of elasticity a reaction is set up and a restoring couple, equal and opposite to the twisting couple is produced. We need to calculate the value of the couple.consider one hollow cylinder of radius *r*and radial thickness *dr.*

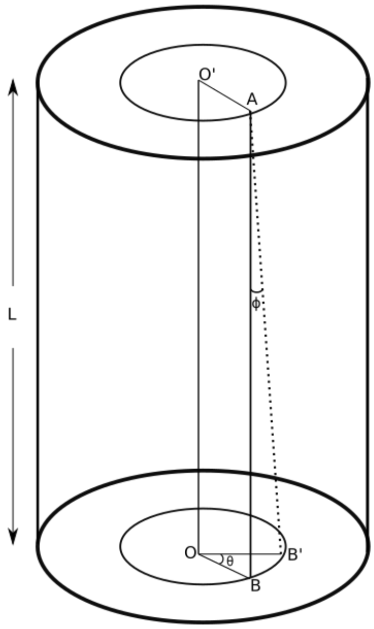
Let OO' be the axis of the cylinder and AB be a line on the hollow cylinder of radius r. On twisting the cylinder through an angle θ, the point B shifts to B^1, the line AB takes up the position AB^1.

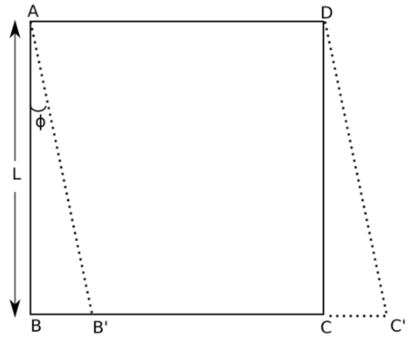
Before twisting, if this hollow cylinder were to be cut along AB and flattened out, it would form a rectangular plate ABCD, but after twisting it takes the shape of a parallelogram AB^1C^1D. The angle through which this hollow cylinder is sheared is $BAB^1=\phi$ and it is the shear produced.

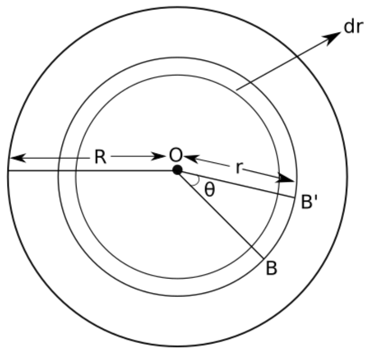
$BB^1=l\phi=x\theta$

$\phi=x\theta/l$

ϕ is maximum at the rim (at x=R) and zero (at x=0) at the axis.







Now, the cross sectional area of the layer under consideration is 2πr *dr*. If 'F' is the shearing force, then the shearing stress T is given by

$$T = \frac{Force}{Area} = \frac{F}{2\pi r dr}$$

$\therefore$ Shearing force F = T(2πrd*r*)

$\therefore$ Rigidity modulus n = Shearing stress/shearing strain.

$$n = \frac{T}{\phi}$$

$$\therefore T = n\phi = \frac{nr\theta}{L}$$

$$\therefore F = \frac{nr\theta}{L}(2\pi r dr) = \frac{2\pi n\theta}{L} r^2 dr$$

The moment of the force about $OO' = \left(\frac{2\pi n\theta}{L} r^2 dr\right) r = \frac{2\pi n\theta}{L} r^3 dr$

This is only for the one layer of the cylinder.

Therefore, twisting couple acting on the entire cylinder $= \int_0^R \frac{2\pi n\theta}{L} r^3 dr$

$$= \frac{2\pi n\theta}{L}\left[\frac{r^4}{4}\right]_0^R$$

$$= \frac{\pi n R^4 \theta}{2L}$$

Couple per unit twist is given by C=Total twisting couple / angle of twist.

$$C = \frac{\pi n R^4 \theta / 2L}{\theta}$$

$$C = \left(\frac{\pi n R^4}{2L}\right)$$

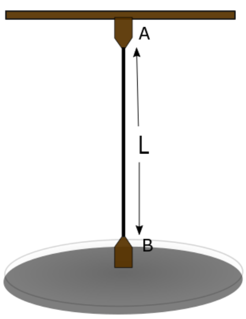
Note: The couple per unit twist is called torsional rigidity.

Torsion Pendulum: A pendulum in which the oscillations are due to the torsion (or twist) is a torsion pendulum.

A Torsional pendulum consists of a heavy metal discsuspended by means of a wire AB of length ‘L’. The top end of the wire is fixed to a rigid support and the bottom end is fixed to the metal disc. When the disc is rotated in a horizontal plane so as to twist the wire, the various elements of the wire undergo shearing strain. The restoring couple of the wire tries to bring the wire back to the original position. Therefore, disc executes torsional oscillations about the mean position.,

Let θ be the angle of twist made by the wire and ‘C’ be the couple per unit twist.

Then the restoring couple per unit twist = Cθ.



At any instant, the deflecting couple (Iα) is equal to the restoring couple, (where I is the moment of inertia of the wire about the axis and α is the angular acceleration).

$$I\left(\frac{d^2\theta}{dt^2}\right) = -C\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

The above relation shows that the angular acceleration is proportional to angular displacement and is always directed towards the mean position. The negative sign indicates that the restoring couple is in the opposite direction to the deflecting couple.
Rearranging the terms of equation (1)

$$\left(\frac{d^2\theta}{dt^2}\right)+\frac{C}{I}\theta = 0 \quad \ldots\ldots(2)$$

Equation (2) indicates that the disc executessimple harmonic motion (SHM).

Therefore, the time period of oscillator is given by relation,

$$T = 2\pi\sqrt{\frac{displacement}{Accleration}} = 2\pi\sqrt{\frac{\theta}{\left(\frac{C}{I} X\ \theta\right)}} = 2\pi\sqrt{\frac{I}{C}}$$

OSCILLATIONS

Introduction:
Motion of bodies can be broadly classified into three categories:

[1] Translational motion
[2] Rotational motion
[3] Vibrational / Oscillatory motion

Translational motion: When the position of a body varies linearly with time, such a motion is termed as translational motion. Example: A car moving on a straight road, a ball moving on the ground.

Rotational motion: When a body as a whole does not change its position linearly with time but rotates about its axis, this motion is said to be rotational motion. Example: rotation of a fly wheel on ball bearings.

Vibrational /Oscillatory motion: When a body executes back and forth motion which repeats over and again about a mean position, then the body is said to have Vibrational/oscillatory motion. If such motion repeats in regular intervals of time then it is called *Periodic motion or Harmonic motion* and the body executing such motion is called **Harmonic oscillator.** In harmonic motion there is a linear relation between force acting on the body and displacement produced. Example: bob of a pendulum clock, motion of prongs of tuning fork, motion of balance wheel of a watch, the up and down motion of a mass attached to a spring.

Note:

1. If there is no linear relation between the force and displacement then the motion is called unharmonic motion. Many systems are unharmonic in nature.

2. In some oscillatory systems the bodies may be at rest but the physical properties of the system may undergo changes in oscillatory manner Examples: Variation of pressure in sound waves, variation of electric and magnetic fields in electromagnetic waves.

Parameters of an oscillatory system

1. **Mean position**: The position of the oscillating body at rest.

2. **Amplitude:** The amplitude of an SHM is the maximum displacement of the body from its mean position.

3. **Time Period:** The time interval during which the oscillation repeats itself is called the time period. It is denoted by **T** and its unit in seconds.

$$\text{Period} = \text{T} = 2\pi \sqrt{\frac{\text{Displacement}}{\text{Acceleration}}}$$

4. **Frequency:** The number of oscillations that a body completes in one second is called the frequency of periodic motion. It is the reciprocal of the time period T and it is denoted by n.

$$\text{Frequency} = \text{n} = \frac{1}{\text{T}}$$

5. **Phase:** It is the physical quantity that expresses the instantaneous position and direction of motion of an oscillating system. If the SHM is represented by y=A sin ($\omega t+\phi$), together with ω as the angular frequency. The quantity ($\omega t+\phi$) of the sine function is called the total phase of the motion at time 't' and 'ϕ' is the initial phase or epoch.

All periodic motions are not vibratory or oscillatory. In this chapter we shall study the simplest vibratory motion along one dimension called **Simple Harmonic Motion** (SHM) which usually occurs in mechanical systems.

SIMPLE HARMONIC MOTION (SHM)

A body is said to be undergoing Simple Harmonic Motion (SHM) when the acceleration of the body is always proportional to its displacement and is directed towards its equilibrium or mean position. Simple harmonic motion can be broadly classified in to two types, namely **linear simple harmonic motion and angular simple harmonic motion.**

Linear Simple Harmonic Motion: If the body executingSHM has a linear acceleration then the motion of the body is linear simple harmonic motion.
Examples: motion of simple pendulum, the motion of a point mass tied with a spring etc.,

Angular Simple Harmonic Motion: If the body executingSHM has an angular acceleration then the motion of the body is angular simple harmonic motion.
Examples: Oscillations of a torsional pendulum

A particle or a system which executes simple harmonic motion is called ***Simple Harmonic Oscillator***. Examples of simple harmonic motion are motion of the bob of a simple pendulum, motion of a point mass fastened to spring, motion of prongs of tuning fork etc.,
Un damped OR Free vibrations
If a body oscillates without the influence of any external force, then the oscillations are called **free oscillations or un damped oscillations**. In free oscillations the body oscillates with its natural frequency and the amplitude remains constant (Fig.1)

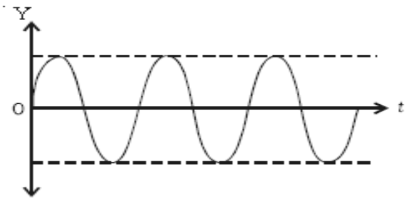


Figure 1: Free Vibrations

In practice it is not possible, actually the amplitude of the vibrating body decreases to zero as a result of friction. Hence, practical examples for free oscillation/vibrations are those in which the friction in the system is negligibly small.

Examples of Simple harmonic oscillator

a) Spring and Mass system

The spring and mass system is an example for linear simple harmonic oscillator. Consider a block of mass *m* suspended from a rigid support through a mass less spring. The restoring force produced in the spring obeys **Hooke's Law**.
According to Hooke's **Law "The restoring force produced in a system is proportional to the displacement".** When the mass is displaced through a distance 'y'and then released, it undergoes SHM. Then the restoring force (F) produced is

$$F = -ky\text{---------} (1)$$

The constant k is called **force constant or spring constant or stiffness constant**. It is a measure of the stiffness of the spring. The negative sign in equation (1) indicates that the restoring force is in the direction of its equilibrium position.

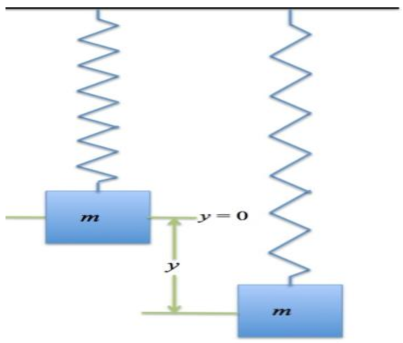


Figure 2: A one-dimensional simple harmonic oscillator.

In equilibrium condition the linear restoring force (F) in magnitude is equal to theweight '*mg*' of the hanging mass, shown in figure 2.

i.e, F = mg $\quad\quad \therefore$ mg = ky

$$\text{and spring constant} \quad k = \frac{mg}{y} --------(2)$$

Now, if the mass is displaced down through a distance 'y' from its equilibrium position and released then it executes oscillatory motion.

$$\text{Velocity of the body} = v = \frac{dy}{dt} \text{ and Acceleration of the body} = a = \frac{d^2y}{d^2t}$$

From Newton's second law of motion for the block we can write

$$F = ma = m\frac{d^2y}{dt^2} ------------(3)$$

from equation (1) substitue F= -ky in equation(3),we get

$$-ky = m\frac{d^2y}{dt^2}$$

$$\text{or } \frac{d^2y}{dt^2} + \frac{k}{m}y = 0$$

$$\text{Substitue } \frac{k}{m} = \omega^2, \quad \text{Where } \omega \text{ is the angular natural frequency.}$$

$$\text{i.e., } \frac{d^2y}{dt^2} + \omega^2 y = 0 ----------(4)$$

Equation (4) is the general differential equation for the *free oscillator*. The mathematical solution of the equation (4), y(t) represents the position as a function of time t. Lety(t) = Asin(ωt+ϕ) be the solution.

$$y(t) = A\sin(\omega t+\phi) --------------------(5)$$

Where A is the ***amplitude*** and (ωt+ϕ)*is* called the ***phase***, ϕ is the initial phase (i.e., phase at t=0).

Period of the oscillator (T)

We have F = ma and also F = -ky. At equilibrium, magnitude of restoring force is equal to weight of hanging mass. i.e, ma = ky

$$\therefore \text{Acceleration} = a = \frac{ky}{m}$$

The period of the oscillator is

$$T = 2\pi\sqrt{\frac{\text{Displacement}}{\text{Acceleration}}} = 2\pi\sqrt{\frac{y}{a}} = 2\pi\sqrt{\frac{y}{\frac{ky}{m}}} = 2\pi\sqrt{\frac{m}{k}}$$

We know that $\omega = \sqrt{\frac{k}{m}}$ and $\frac{1}{\omega} = \sqrt{\frac{m}{k}}$, rewrite period expression

Time Period $T = \frac{2\pi}{\omega}$, frequency $n = \frac{1}{T} = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$

Although the position and velocity of the oscillator are continuously changing, the total energy **E**remains constant and is given by

$$E = \frac{1}{2}mv^2 + \frac{1}{2}ky^2 \text{ --------- } (6)$$

The two terms in (6) are the kinetic and potential energies, respectively.

Velocity and acceleration of SHM

We can find velocity from the expression for displacement, $y = A\sin(\omega t + \phi)$

Velocity in SHM

The rate of change of displacement is the velocity of the vibrating particle. By differentiating the above expression with time, we get

$$v = \frac{dy}{dt} = A\omega\cos(\omega t + \phi)$$

$$v = A\omega\sqrt{1 - \sin^2(\omega t + \phi)} \quad \text{As} \quad \sin(\omega t + \phi) = \frac{y}{A}$$

$$v = A\omega\sqrt{1 - \frac{y^2}{A^2}}$$

$$\therefore v = \omega\sqrt{A^2 - y^2}$$

This is the expression for velocity of the particle at any displacement 'y'. The maximum velocity is obtained by substituting y =0, $\therefore v_{max} = \omega A$

Since y = 0 corresponds to its mean position, the particle has maximum velocity when it is at its mean position. At the maximum displacement, i.e., at the extreme position y =A of the particle, the velocity is zero.

Acceleration in SHM

The rate of change of velocity is the acceleration of the vibrating particle. Differentiating velocity expression with respect to time t, we get acceleration

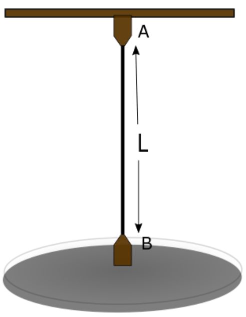
$$a = \frac{d^2y}{dt^2} = -A\omega^2\sin(\omega t + \phi)$$

$$a = \frac{d^2y}{dt^2} = -\omega^2 y$$

The above equation gives acceleration of the oscillating particle at any displacement. This equation is the standard equation of SHM.

Since y =0 corresponds to its mean position, the particle has minimum acceleration. At the maximum displacement, i.e., at the extreme position y=A, maximum acceleration is $-\omega^2 A$

Torsion Pendulum: A pendulum in which the oscillations are due to the torsion (or twist) is a torsion pendulum.
A Torsional pendulum consists of a heavy metal disc suspended by means of a wire AB of length 'L'. The top end of the wire is fixed to a rigid support and the bottom end is fixed to the metal disc. When the disc is rotated in a horizontal plane so as to twist the wire, the various elements of the wire undergo shearing strain. The restoring couple of the wire tries to bring the wire back to the original position. Therefore, disc executes torsional oscillations about the mean position.,
Let θ be the angle of twist made by the wire and 'C' be the couple per unit twist.
Then the restoring couple per unit twist = Cθ.



At any instant, the deflecting couple (Iα) is equal to the restoring couple, (where I is the moment of inertia of the wire about the axis and α is the angular acceleration).
I = -Cθ ……………..(1)

The above relation shows that the angular acceleration is proportional to angular displacement and is always directed towards the mean position. The negative sign indicates that the restoring couple is in the opposite direction to the deflecting couple.
Rearranging the terms of equation (1)

+C/I θ=0 …..(2)

Equation (2) indicates that the disc executes simple harmonic motion (SHM).

Therefore, the time period of oscillator is given by relation,
Note: For a given wire of length l, radius r and rigidity modulus η

$$\text{Torsional constant of the wire } C = \frac{\pi \eta r^4}{2l}$$

Damped vibrations:

Free vibrations are oscillations in which the friction/resistance considered is zero or negligible. Therefore, the body will keep on vibrating indefinitely with respect to time. In real sense if a body set into vibrations, its amplitude will be continuously decreasing due to friction/resistance and so the vibrations will die after some time. Such vibrations are called ***damped vibrations*** (Fig. 6).

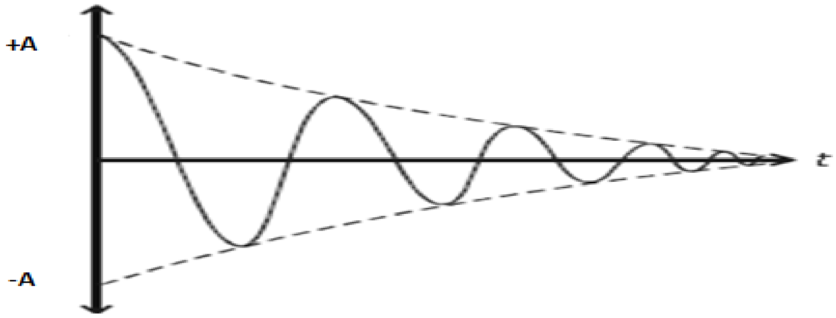


Figure 6: Damped vibrations

Differential equation of damped vibrations and solution

A block of mass *m* connected to the free end of a spring is partially immersed in a liquid (Fig. 7) and is subjected to small vibrations. The damping force encountered is more when the block moves in the liquid and hence the amplitude of vibration decreases with time and ultimately stops vibrating; this is due to energy loss from viscous forces.

Let y to be the displacement of the body from the equilibrium state at any instant of time'‘t’, then dy/dt is the instantaneous velocity.

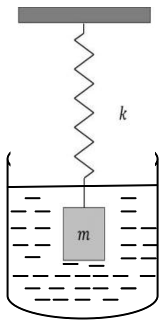


Figure 7: Vertical spring and mass system

The two forces acting on the body at this instant are:

i. A restoring force which is proportional to displacement and acts in the opposite direction, it may be written as

$$F_{restoring} = -ky$$

ii. A *frictional or damping* force is directly proportional to the velocity and is oppositely directed to the motion, it may be written as

$$F_{damping} = -r\frac{dy}{dt}$$

The net force on an oscillator subjected to a linear damping force that is linear in velocity is simply the sum is thus given by

$$\mathrm{F} = \mathrm{F}_{\mathrm{restoring}} + \mathrm{F}_{\mathrm{damping}} = -\mathrm{ky} - \mathrm{r}\frac{\mathrm{dy}}{\mathrm{dt}}$$

But by Newton's law of motion, $\mathrm{F} = \mathrm{m}\frac{\mathrm{d}^2\mathrm{y}}{\mathrm{dt}^2}$

where m is the mass of the body and $\frac{\mathrm{d}^2\mathrm{y}}{\mathrm{dt}^2}$ is the accleration of the body.

$$\text{Then, } \mathrm{m}\frac{\mathrm{d}^2\mathrm{y}}{\mathrm{dt}^2} = -\mathrm{ky} - \mathrm{r}\frac{\mathrm{dy}}{\mathrm{dt}}$$

$$\text{or} \quad \frac{\mathrm{d}^2\mathrm{y}}{\mathrm{dt}^2} + \frac{\mathrm{r}}{\mathrm{m}}\frac{\mathrm{dy}}{\mathrm{dt}} + \frac{\mathrm{k}}{\mathrm{m}}\mathrm{y} = 0 \text{-----------(1)}$$

Let $\frac{r}{m}$ =2b and $\frac{k}{m} = \omega^2$, then the above equation takes the form,

$$\frac{d^2y}{dt^2} + 2b\frac{dy}{dt} + \omega^2 \mathrm{y} = 0 \text{ -----------------(2)}$$

This is the differential equation of second order. In order to solve this equation, we assume its solution as

$$\mathrm{y} = \mathrm{A}e^{\alpha t} \text{ --------------------(3)}$$

Where A and α are2 arbitrary constants (variational parameters) . Differentiating equation (3) with respect to time t, we have

$$\frac{\mathrm{dy}}{\mathrm{dt}} = \mathrm{A}\alpha e^{\alpha t} \text{ and } \frac{\mathrm{d}^2 y}{\mathrm{dt}^2} = \mathrm{A}\alpha^2 e^{\alpha t}$$

By substituting these values in equation (2), we have

$$A\alpha^2 e^{\alpha t} + 2bA\alpha e^{\alpha t} + \omega^2 \mathrm{A}e^{\alpha t} = 0$$

or

$$Ae^{\alpha t}\left(\alpha^2 + 2b\alpha + \omega^2\right) = 0$$

For the above equation to be satisfied, either y=0, or $\left(\alpha^2 + 2b\alpha + \omega^2\right) = 0$

Since y=0, corresponds to a trivial solution, one has to consider the solution

$$\left(\alpha^2 + 2b\alpha + \omega^2\right) = 0$$

Thestandard solution of the above quadratic equation has two roots, it is given by,

$$\alpha = \frac{-2b^2 \pm \sqrt{4b^2 - 4\omega^2}}{2}$$

$$\alpha = -b \pm \sqrt{b^2 - \omega^2}$$

Therefore, the general solution of equation (2) is given by

$$y = C\,e^{(-b + \sqrt{b^2-\omega^2})t} + D\,e^{(-b - \sqrt{b^2-\omega^2})t} -----(4)$$

Where C and D are constants, the actual solution depends upon whether $b^2 > \omega^2$ **,** $b^2 = \omega^2$ and $b^2 < \omega^2$**.**

Case 1: Heavy damping ($b^2 > \omega^2$)

In this case, $\sqrt{b^2 - \omega^2}$ is real and less than b, therefore in equation (4) both the exponents are negative. It means that the displacement 'y' of the particle decreases continuously with time. That is, the particle when once displaced returns to its equilibrium position quite slowly without performing any oscillations (Fig.8). Such a motion is called **'*overdamped*'**or**'*aperiodic*'**motion. This type of motion is shown by a pendulum moving in thick oil or by a dead beat moving coil galvanometer.

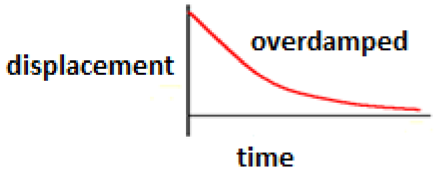


Figure 8: Heavy damping or overdamped motion

Case 2: Critical damping ($b^2 = \omega^2$)

By substituting $b^2 = \omega^2$ in equation (4) the solution does not satisfy equation (2).Hence we consider the case when $\sqrt{b^2 - \omega^2}$ is not zero but is a very small quantity β. The equation (4) then can be written as

$$y = C\,e^{(-b+\beta)t} + D\,e^{(-b-\beta)t}$$

$$y = e^{-bt}\,(C\,e^{\beta t} + D\,e^{-\beta t})$$

$$\text{since } \beta \text{ is small, we can approximate,}$$

$$e^{\beta t} = 1 + \beta t \; and \; e^{-\beta t} = 1 - \beta t, \text{ on the basis of exponential series expansion}$$

$$y = e^{-bt}\left[C(1+\beta t) + D(1-\beta t)\right]$$

$$y = e^{-bt}\left[(C+D) + \beta t\,(C-D)\right] \text{ which is the product of terms}$$

As't' is present in e^{-bt} and also in the term $\beta t\,(C-D)$, both of them contribute to the variation of y with respect to time. But by the virtue of having t in the exponent, the term e^{-bt} predominantly contributes to the equation. Only for small values of t, the term

$\beta t(C-D)$ contributesin a magnitude comparable to that of e^{-bt}. Therefore, though y decreases throughout with increase of t, the decrement is slow in the beginning, and then decreases rapidly to approach the value zero. i.e., the body attains equilibrium position (Fig.9). Such damping motion of a body is called **critical damping**. This type of motion is exhibited by many pointer instruments such as voltmeter, ammeter...etc in which the pointer moves to the correct position and comes to rest without any oscillation.

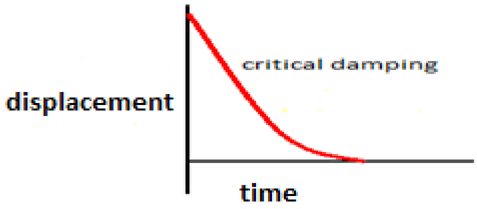


Figure 9: Critical damping motion.

Case 3: Low damping ($b^2 < \omega^2$)

This is the actual case of damped harmonic oscillator.In this case $\sqrt{b^2-\omega^2}$ is imaginary. Let us write

$$\sqrt{b^2-\omega^2} = i\sqrt{\omega^2-b^2} = i\beta'$$

where $\beta' = \sqrt{\omega^2-b^2}$ and $i = \sqrt{-1}$.

Then, equation (4) becomes

$$y = Ce^{(-b+i\beta')t} + De^{(-b-i\beta')t}$$

$$y = e^{-bt}(Ce^{i\beta't} + De^{-i\beta't})$$

$$y = e^{-bt}(C(\cos\beta't + i\sin\beta't) + D(\cos\beta't - i\sin\beta't))$$

$$y = e^{-bt}\left[(C+D)\cos\beta't + i(C-D)\sin\beta't)\right]$$

Rewriting $C+D = A\sin\phi$ and $i(C-D) = A\cos\phi$,

Where A and ϕ are constants.

$$y = e^{-bt}\left[A\sin\phi\cos\beta't + A\cos\phi\sin\beta't\right]$$

$$y = Ae^{-bt}\sin(\beta't+\phi)$$

This equation $y = Ae^{-bt}\sin(\beta't+\phi)$ represents the **damped harmonic oscillations**. The oscillations are not simple harmonic because the amplitude (Ae^{-bt}) is not constant and decreases with time (t). However, the decay of amplitude depends upon the damping factor b. This motion is known as **under damped motion** (Fig.10). The motion of pendulum in air and the motion of ballistic coil galvanometer are few of the examples of this case.

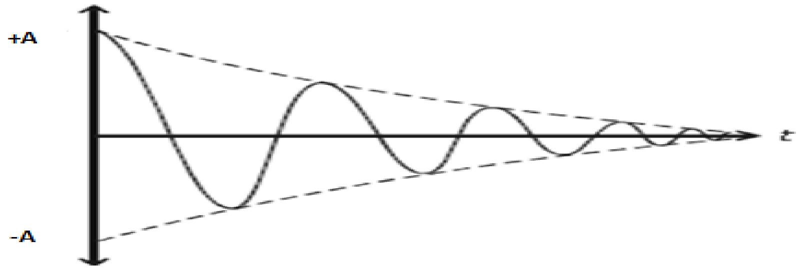


Figure 10: Under damped motion

The time period of damped harmonic oscillator is given by

$$T=\frac{2\pi}{\beta'}=\frac{2\pi}{\sqrt{\omega^2-b^2}}\frac{2\pi}{\sqrt{\frac{k}{m}-\frac{r^2}{4m^2}}}$$

The frequency of damped harmonic oscillator is given by

$$n=\frac{1}{T}=\frac{1}{2\pi}\sqrt{\frac{k}{m}-\frac{r^2}{4m^2}}$$

Forced Vibrations:

In the case of damped vibrations, the amplitude of vibrations decrease with the time exponentially due to dissipation of energy and the body eventually comes to a rest. When a body experiences vibrations due to the influence of an external driving force the body can continue its vibration without coming to a rest. Such vibrations are called ***forced vibrations***.

For example: When a tuning fork is struck on a rubber pad and its stem is placed on a table, the table is set in vibrations with the frequency of the fork. These oscillations of the table are the **forced oscillations/vibrations**. So forced vibrations can also be defined as the vibrations in which the body vibrates with frequency other than natural frequency of the body, and they are due to applied external periodic force.

DIFFERENTIAL EQUATION OF FORCED VIBRATIONS AND SOLUTION

Suppose a particle of mass m is connected to a spring. When it is displaced and released it starts oscillating about a mean position. The particle is driven by external periodic force $(F_o \sin\omega_d t)$. The oscillations experiences different kinds of forces viz,

1) A restoring force proportional to the displacement but oppositely directed, is given by $F_{restoring}$= -ky, where k is known as force constant.

2) A frictional or damping force proportional to velocity but oppositely directed, is given by

$$F_{damping} = -r\frac{dy}{dt}$$, where r is the frictional force/unit velocity.

3) The applied external periodic force is represented by $(F_o \sin\omega_d t)$, where F_o is the maximum value of the force and ω_d is the angular frequency of the driving force.

The total force acting on the particle is given by,

$$F_{net} = F_{external} + F_{restore} + F_{damp} = F_o \sin\omega_d t - r\frac{dy}{dt} - ky$$

By Newton's second law of motion $F = m\frac{d^2y}{dt^2}$.

Hence, $m\frac{d^2y}{dt^2} = F_o \sin\omega_d t - r\frac{dy}{dt} - ky$

The differential equation of forced vibrations is given by

$$\frac{d^2y}{dt^2} + \frac{r}{m}\frac{dy}{dt} + \frac{k}{m}y = \frac{F_o \sin\omega_d t}{m} \text{------------}(1)$$

Substitute $\frac{r}{m} = 2b$, $\frac{k}{m} = \omega^2$ and $\frac{F_o}{m} = f$, then equation (1) becomes

$$\frac{d^2y}{dt^2} + 2b\frac{dy}{dt} + \omega^2 y = f \sin\omega_d t \text{------------}(2)$$

Where b is the damping constant,
F_o is the amplitude of the external driving force
and ω_d is the angular frequency of the external force.

The solution of the above equation (2) is given by

$$y = A\sin(\omega_d t - \phi) \text{---------}(3)$$

Where, A is the amplitude of the forced vibrations.

By differentiating equation (3) twice with respect to time t, we get

$$\frac{dy}{dt} = \omega_d A\cos(\omega_d t - \phi) \quad and \quad \frac{d^2y}{dt^2} = -\omega_d^2 A\sin(\omega_d t - \phi)$$

By substituting the values of y, dy/dt and d^2y/dt^2 inequation (2), we get

$$-\omega_d^2 A\sin(\omega_d t - \phi) + 2b\,\omega_d A\cos(\omega_d t - \phi) + \omega^2 A\sin(\omega_d t - \phi) = f\sin\{(\omega_d t - \phi) + \phi\}$$

or

$$A(\omega^2 - \omega_d^2)\sin(\omega_d t - \phi) + 2b\omega_d A\cos(\omega_d t - \phi) = f\sin(\omega_d t - \phi)\cos\phi + f\cos(\omega_d t - \phi)\sin\phi \text{------}(4)$$

If equation (4) holds good for all values of t, then the coefficients of sin (ω_dt- φ) and cos(ω_dt- φ) must be equal on both sides, then

$$A(\omega^2 - \omega_d^2) = f\cos\phi \text{----------}(5)$$

$$and \qquad 2b\omega_d A = f\sin\phi \text{----------}(6)$$

By squaring and adding equations (5) and (6), we have

$$A^2(\omega^2-\omega_d^2)^2+4b^2\omega_d^2A^2=f^2$$

$$A=\frac{f}{\sqrt{(\omega^2-\omega_d^2)^2+4b^2\omega_d^2}}--------(7)$$

By dividing equation (6) by (5), we get

$$\tan\phi=\frac{2b\omega_d}{\omega^2-\omega_d^2} \quad , \quad \text{Phase } \phi=\tan^{-1}\left[\frac{2b\omega_d}{\omega^2-\omega_d^2}\right]------(8)$$

Substituting the value of A in equation (3) we get

$$y=\frac{f}{\sqrt{(\omega^2-\omega_d^2)^2+4b^2\omega_d^2}}\sin(\omega_d t-\phi)\text{---------------}(9)$$

This is the solution of the differential equation of the **Forced Harmonic Oscillator**. From equations (7) and (8),it is clear that the amplitude and phase of the forced oscillations depend upon $(\omega^2-\omega_d^2)$, i.e., these depend upon the driving frequency (ω_d) and the natural frequency (ω) of the oscillator.

We shall study the behavior of amplitude and phase in three different stages of frequencies i.e, ***low frequency, resonant frequency and high frequency*.**

Case I: When driving frequency is low i.e., $(\omega_d << \omega)$. In this case, amplitude of the Vibrations is given by

$$\mathrm{A}=\frac{\mathrm{f}}{\sqrt{\left(\omega^2-\omega_\mathrm{d}^2\right)^2+4\mathrm{b}^2\omega_\mathrm{d}^2}}$$

As $\omega_\mathbf{d} << \omega$, $\quad \sqrt{\left(\omega^2-\omega_\mathrm{d}^2\right)^2}=\sqrt{\left(\omega^2\right)^2}=\omega^2$ and $4\mathrm{b}^2\omega_\mathrm{d}^2=0 \ \because \omega_\mathrm{d}\rightarrow 0$

$$\mathrm{A}=\frac{\mathrm{f}}{\omega^2}=\frac{\mathrm{F_0}/\mathrm{m}}{\mathrm{k}/\mathrm{m}}=\frac{\mathrm{F_0}}{\mathrm{k}} \qquad \because \mathrm{f}=\frac{\mathrm{F_0}}{\mathrm{m}} \text{ and } \omega^2=\frac{\mathrm{k}}{\mathrm{m}}$$

$$\textbf{and} \quad \varphi=\tan^{-1}\left[\frac{2\mathbf{b}\boldsymbol{\omega}_\mathbf{d}}{\left(\boldsymbol{\omega}^2-\boldsymbol{\omega}_\mathbf{d}^2\right)}\right]=\tan^{-1}(\mathbf{0})=\mathbf{0}$$

This shows that the amplitude of the vibration is independent of frequency of the driving force and is dependent on the magnitude of the driving force and the force-constant (k). In such a case the force and displacement are always in phase.

Case II: When $\omega_d = \omega$ i.e., frequency of the driving force is equal to the natural frequency of the body. This frequency is called resonant frequency. In this case, amplitude of vibrations is given by

$$A = \frac{f}{\sqrt{(\omega^2 - \omega_d^2)^2 + 4b^2\omega_d^2}}$$

$$A = \frac{f}{2b\omega_d} = \frac{F_o/m}{(r/m)\omega_d} = \frac{F_o}{r\omega_d}$$

$$and\ \phi = \tan^{-1}\left[\frac{2b\omega_d}{\omega^2 - \omega_d^2}\right] = \tan^{-1}\left[\frac{2b\omega_d}{0}\right] = \tan^{-1}(\infty) = \frac{\pi}{2}$$

Under this situation, the amplitude of the vibrations becomes maximum and is inversely proportional to the damping coefficient. For small damping, the amplitude is large and for large damping, the amplitude is small. The displacement lags behind the force by a phase $\pi/2$.

Case III: When $(\omega_d >> \omega)$ i.e., the frequency of force is greater than the natural frequency of the body. In this case, amplitude of the vibrations is given by

$$\mathrm{A} = \frac{\mathrm{f}}{\sqrt{\left(\omega^2 - \omega_\mathrm{d}^2\right)^2 + 4\mathrm{b}^2\omega_\mathrm{d}^2}}$$

$$\omega_\mathrm{d} >> \omega,\ \therefore \left(\omega^2 - \omega_\mathrm{d}^2\right)^2 = \omega_d^4 \text{ since } \omega_\mathrm{d} \text{ is very large, } \omega_d^4 >>> 4\mathrm{b}^2\omega_\mathrm{d}^2$$

$$\therefore \mathrm{A} = \frac{\mathrm{f}}{\omega_\mathrm{d}^2} = \frac{\mathrm{F_0}/\mathrm{m}}{\omega_d^2} = \frac{\mathrm{F_0}}{\mathrm{m}\omega_d^2}$$

$$\text{and}\ \phi = \tan^{-1}\left[\frac{2\mathrm{b}\omega_\mathrm{d}}{\left(\omega^2 - \omega_\mathrm{d}^2\right)}\right] = \tan^{-1}\left[-\frac{2b}{\omega_d}\right] = \tan^{-1}(-0) = \pi$$

This shows that amplitude depends on the mass and continuously decreases as the driving frequency ω_d is increased and phase difference towards π.

Resonance:

If we bring a vibrating tuning fork near another stationary tuning fork of the same natural frequency as that of vibrating tuning fork, we find that stationary tuning fork also starts vibrating. This phenomenon is known asResonance**.**

Resonance is a phenomenon in which a body vibrates with its natural frequency with maximum amplitude under the influence of an external vibration with the same frequency.

Theory of resonant vibrations:

(a) Condition of amplitude resonance. In case of forced vibrations, the expression for amplitude A and phase ϕ is given by,

$$A = \frac{f}{\sqrt{(\omega^2 - \omega_d^2)^2 + 4b^2\omega_d^2}} \text{ and } \phi = \tan^{-1}\left[\frac{2b\omega_d}{\omega^2 - \omega_d^2}\right]$$

The amplitude expression shows the variation with the frequency of the driving force ω_d. For a particular value of ω_d, the amplitude becomes maximum. The phenomenon of amplitude becoming a maximum is known as amplitude resonance. The amplitude is maximum when $\sqrt{(\omega^2-\omega_d^2)^2+4b^2\omega_d^2}$ is minimum.If the damping is small i.e., b is small, the condition of maximum amplitude reduced to $A_{max}=\frac{f}{2b\omega_d}$.

(b) Sharpness of the resonance.

We have seen that the amplitude of the forced oscillations is maximum when the frequency of the applied force is at resonant frequency. If the frequency changes from this value, the amplitude falls. When the fall in amplitude for a small change from the resonance condition is very large, the resonance is said to be sharp and if the fall in amplitude is small, the resonance is termed as flat. Thus the term sharpness of resonance can be defined as the rate of fall in amplitude, with respect to the change in forcing frequency on either side of the resonant frequency.

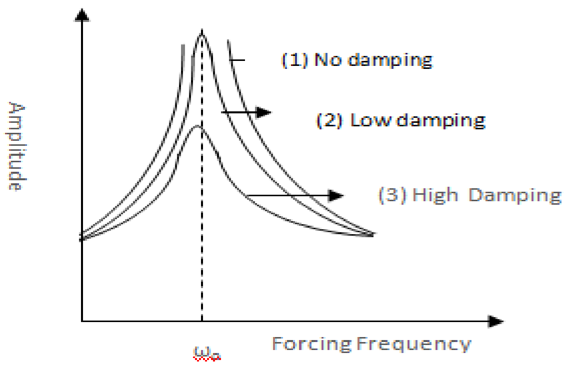


Figure 11: Effect of damping on sharpness of resonance

Figure 11, shows the variation of amplitude with forcing frequency for different amounts of damping. Curve (1) shows the variation of amplitude when there is no damping i.e., b=0. In this case the amplitude is infinite at $\omega_d=\omega$. This case is never realized in practice due to friction/dissipation forces, as a slight damping factor is always present. Curves (2) and (3) show the variation of amplitude with respect to low and high damping. It can be seen that the resonant peak moves towards the left as the damping factor is increased. It is also observed that the value of amplitude, which is different for different values of b (damping), diminishes as the value of b increases. This indicates that the smaller is the damping, sharper is the resonance or large is the damping, flatter is the resonance.

Example for Electrical Resonance:LCR circuit

An L-C-R circuit fed by an alternating emf is a classic example for a forced harmonic oscillator. Consider an electric circuit containing an inductance L, capacitance C and resistance R in series as shown in the figure 13. An alternating emf has been applied to a circuit is represented by $E_o \sin\omega_d t$.

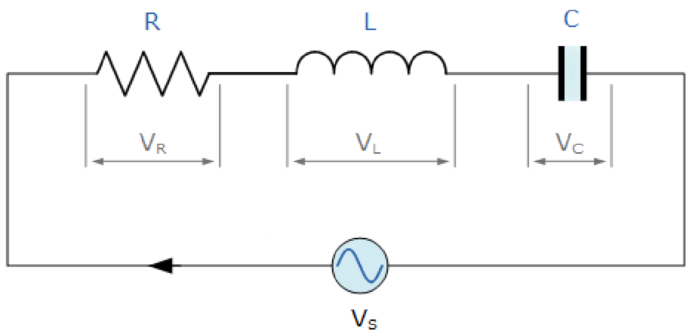


Figure 13: Series LCR circuit.

Let q be the charge on the capacitor at any instant and I be the current in the circuit at any instant. The potential difference across the capacitor is $\frac{q}{C}$, the back emf due to self inductance in the inductor is $L\frac{dI}{dt}$ and, the potential drop across the resistor is IR. The sum of voltages across the three LCR elements illustrated must be equal the voltage supplied by the source element. Hence the voltage equation at any instant is given by,

$$V_L + V_R + V_C = V_S(t)$$

$$L\frac{dI}{dt} + IR + \frac{q}{C} = E_o \sin\omega_d t$$

Differentiating, we get

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}\frac{dq}{dt} = \omega_d E_o \cos\omega_d t$$

$$\text{but } \frac{dq}{dt} = I, \ \therefore \quad L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{I}{C} = \omega_d E_o \cos\omega_d t$$

$$\frac{d^2I}{dt^2} + \left(\frac{R}{L}\right)\frac{dI}{dt} + \left(\frac{1}{LC}\right)I = \left(\frac{\omega_d E_o}{L}\right)\cos\omega_d t$$

$$\frac{d^2I}{dt^2} + \left(\frac{R}{L}\right)\frac{dI}{dt} + \left(\frac{1}{LC}\right)I = \left(\frac{\omega_d E_o}{L}\right)\sin\left(\omega_d t + \frac{\pi}{2}\right) ----(1)$$

This is the differential equation of the ***forced oscillations*** in the electrical circuit. It is similar to equationof motion of a mechanical oscillator driven by an external force.

$$\frac{d^2y}{dt^2} + 2b\frac{dy}{dt} + \omega^2 y = \frac{F_o}{m}\sin\omega_d t \text{ ---------------------------(2)}$$

The explicit and preciseconnection with the mechanical oscillation equation is given below:

Displacement:	**y**	$\longleftrightarrow$	**I**
Velocity :	$\frac{dy}{dt}$	$\longleftrightarrow$	$\frac{dI}{dt}$
Damping constant:	**2b**	$\longleftrightarrow$	**R/L**
Natural frequency:	ω	$\longleftrightarrow$	$1/\sqrt{LC}$
External force :	**F(t)**	$\longleftrightarrow$	E_0
And	F_0/m	$\longleftrightarrow$	$\omega_d E_0/m$

Forced oscillations (refer previous section)	LCR Circuit application
Starting with equation from forced vibrations, we have $$\frac{d^2y}{dt^2} + 2b\frac{dy}{dt} + \omega^2 y = \frac{F_o}{m}\sin\omega_d t$$ Amplitude of mechanical vibrations is given by $$A = \frac{F_o/m}{\sqrt{(\omega^2-\omega_d^2)^2+4b^2\omega_d^2}}$$	The equivalent equation for LCR circuit is given by $$\frac{d^2I}{dt^2}+\left(\frac{R}{L}\right)\frac{dI}{dt}+\left(\frac{1}{LC}\right)I=\left(\frac{\omega_d E_o}{L}\right)\sin\left(\omega_d t+\frac{\pi}{2}\right)$$ Amplitude of current I_o is given by $$I_O=\frac{\frac{\omega_d}{L}E_o}{\sqrt{(\omega^2-\omega_d^2)^2+4b^2\omega_d^2}}$$ Substitute for $\omega^2=\frac{1}{LC}$ and $4b^2=\frac{R^2}{L^2}$, we get $$I_o=\frac{\frac{\omega_d}{L}E_o}{\sqrt{\left(\frac{1}{LC}-\omega_d^2\right)^2+\frac{R^2}{L^2}\omega_d^2}}$$ In the denominator, multiply and divde the term $\frac{1}{LC}$ by ω_d and ω_d^2 by L, we get $$I_o=\frac{\frac{\omega_d}{L}E_o}{\sqrt{\left[\left(\frac{1}{LC}\frac{\omega_d}{\omega_d}\right)-\left(\frac{L}{L}\omega_d^2\right)\right]^2+\frac{R^2}{L^2}\omega_d^2}}$$ $$I_o=\frac{\frac{\omega_d}{L}E_o}{\sqrt{\frac{\omega_d^2}{L^2}\left[\left(\frac{1}{C\omega_d}\right)-(L\omega_d)\right]^2+\frac{R^2}{L^2}\omega_d^2}}$$

	$I_o = \dfrac{\frac{\omega_d}{L} E_o}{\frac{\omega_d}{L}\sqrt{\left(\frac{1}{C\omega_d} - L\omega_d\right)^2 + R^2}}$ substitute $\frac{1}{C\omega_d} = X_C$ (Capacitive reactance) and $L\omega_d = X_L$ (Inductive reactance) $I_o = \dfrac{E_o}{\sqrt{(X_C - X_L)^2 + R^2}} = \dfrac{E_o}{\sqrt{[(X_L - X_C)]^2 + R^2}}$

The solution of the equation (1) for the current at any instant in the circuit is of the form

$$I = \frac{E_o}{\sqrt{R^2 + \left(\omega_d L - \frac{1}{\omega_d C}\right)^2}} \sin(\omega_d t - \phi) = I_o \sin(\omega_d t - \phi) ------(3)$$

Electrical Impedance: The ratio of amplitudes of alternating emf and current in ac circuit is called the electrical impedance of the circuit. *i.e.,* $Z = \frac{E_o}{I_o} \quad \therefore\ I_o = \frac{E_o}{Z}$

The amplitude of the current is $I_o = \dfrac{E_o}{\sqrt{R^2 + \left(\omega_d L - \frac{1}{\omega_d C}\right)^2}} = \dfrac{E_o}{Z}$

Impedance of the circuit $Z = \sqrt{R^2 + \left(\omega_d L - \frac{1}{\omega_d C}\right)^2}$ ---------(4).

The quantity $\left(\omega_d L - \frac{1}{\omega_d C}\right)$ is the net reactance of the circuit which is the difference between the inductive reactance $(X_L) = \omega_d L$ and the capacitive reactance $(X_C) = \frac{1}{\omega_d C}$. Equation (3) shows that the current I lags in phase with the applied emf $E_o \sin\omega_d t$ by an angle ϕ and is given by $\phi = \tan^{-1}\left(\dfrac{\omega_d L - \frac{1}{\omega_d C}}{R}\right) = \tan^{-1}\left(\dfrac{X_L - X_C}{R}\right)$

The following three cases arise:

1. When $X_L > X_C$, ϕ is positive, that is, the current lags behind the emf. Circuit is inductive.

2. When $X_L < X_C$, ϕ is negative, that is, the current leads behind the emf. Circuit is capacitive.

3. When $X_L = X_C$, ϕ is zero, that is, the current is in phase with the emf. Circuit is resistive.

Electrical Resonance: According to equation (4), the current have its maximum amplitude when

$$X_L - X_C = 0 \;\; or \;\; \omega_d L - \frac{1}{\omega_d C} = 0 \; or \; \omega_d L = \frac{1}{\omega_d C} \quad or \; \omega_d = \frac{1}{\sqrt{LC}}$$

Where ω_d is the angular frequency of the applied emf, while $\omega = \frac{1}{\sqrt{LC}}$ is the (angular) natural frequency of the circuit.
Hence, the maximum amplitude of the current oscillations occurs when the frequency of the applied emf is exactly equal to the natural (un-damped) frequency of the electrical circuit. This is the condition of **electrical resonance**.

Sharpness of resonance and Bandwidth:
When an alternating emf is applied to an LCR circuit, electrical oscillations occur in the circuit with the frequency $\omega_d$is equal to the applied emf. The amplitude of these oscillations (current amplitude) in the circuit is given by

$$I_o = \frac{E_o}{\sqrt{R^2 + \left(\omega_d L - \frac{1}{\omega_d C}\right)^2}} = \frac{E_o}{Z}, \;\; \text{Where } Z \text{ is the impedance of the circuit.}$$

At resonance, when the frequency ω_d of the applied emf is equal to the natural frequency $\omega = \frac{1}{\sqrt{LC}}$ of the circuit, the current amplitude I_o is maximum and is equal to E_o/R. Thus at resonance the impedance Z of the circuit is R. At other values of ω_d, the current amplitude I_o is smaller and the impedance Z is larger than R.

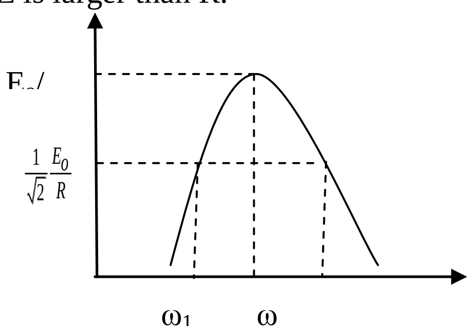


Figure 14: Graphical plot of current vs frequency.

The variation of the current amplitude I_o with respect to applied emf frequency ω_d is shown in the figure 14. $I_o$attains maximum value (E_o/R) when ω_d has resonant value ω and decreases as ω_d changes from ω. The rapidity with which the current falls from its resonant value (E_o/R) with change in applied frequency is known as the **sharpness of resonance.** It is measured by the ratio of the resonant frequency $\omega$to the difference of two frequencies ω_1 and $\omega_2$taken at $\frac{1}{\sqrt{2}}$ of the resonant (ω) value.

i.e**.,** **Sharpness of resonance (Q)** $= \dfrac{\omega}{\omega_2 - \omega_1}$,

ω_1 and ω_2 are known as the **half power frequencies**.

The difference of half power frequencies, ω_1 - ω_2 is known as **"band-width"**. The smaller is the bandwidth, the sharper is the resonance.

SI No	Sample Questions	CO
	Elasticity	
1.	What is stress, strain ?	1
2.	What is Poisson's ratio ?	1
3.	What is a cantilever ? Define neutral layer of a cantilever .	1
4.	Why the Poisson's ratio cannot be negative ?	2
5.	The restoring force on a layer above the neutral layer points inwards, while that for a layer below the neutral layer points outwards. Explain.	2
6.	Define Youngs's Modulus and discuss 2 examples where it is applicable.	2
7.	Define Rigidity Modulus and discuss 2 examples where it is applicable.	2
8.	Define Bulk Modulus and discuss 2 examples where it is applicable.	2
	Oscillations	2
9.	What is simple harmonic motion?	1
10.	Write the general equation representing SHM.	1
11.	List any two characteristics of SHM.	1
12.	A particle executes a S.H.M. of period 10 seconds and amplitude of 2 meter. Calculate its maximum velocity.	3
13.	Hydrogen atom has a mass of 1.68×10^{-27} kg, when attached to a certain massive molecule it oscillates as a classical oscillator with a frequency10^{14} cycles per second and with amplitude of 10^{-10} m. Calculate the acceleration of the oscillator.	3
14.	A body executes S.H.M such that its velocity at the mean position is 4cm/s and its amplitude is 2cm. Calculate its angular velocity.	3
15.	What is free vibration?	1
16.	What is damped vibration?	1
17.	Give any two examples for damped vibration.	1
18.	What is forced vibration?	1
19.	Given two vibrating bodies what is the condition for obtaining resonance?	2
20.	Explain why a loaded bus is more comfortable than an empty bus?	2

21.	What is a simple harmonic oscillator?	1
22.	What is torsional oscillation?	1
23.	Displacement of a particle of mass 10g executes SHM given by $x = 15sin\left(\frac{2\pi t}{T} + \varphi\right)$ and its displacement at t=0 is 3cm where the amplitude isn15cm. Calculate the initial phase of the particle.	3
24.	Name the two forces acting on a system executing damped vibration.	2
25.	How critical damping is beneficiary in automobiles?	1
26.	What is restoring force?	1
	Descriptive Questions	
	Elasticity	
1.	With a neat labelled diagram derive a condition for the dynamical equilibrium between the restoring moment and bending moment of a cantilever loaded at the free end.	2
2.	Starting from the moment balance equation, derive an expression for the mid-point depression of a cantilever which is loaded at the free end.	2
3.	Derive an expression for the couple per unit twist of a wire of rigidity modulus η, length L and radius r.	2
	Oscillations	
1.	Every SHM is periodic motion but every periodic motion need not be SHM. Why? Support your answer with an example.	2
2.	Distinguish between linear and angular harmonic oscillator?	1
3.	Setup the differential equation for SHM.	2
4.	Define the terms (i) time period (ii) frequency (iii) phaseand (iii) angular frequency of oscillations	1
5.	What is the phase difference between (i) velocity and acceleration (ii) acceleration and displacement of a particle executing SHM?	2
6.	Show graphically the variation of displacement, velocity and acceleration of a particle executing SHM.	1
7.	Explain the oscillations of a mass attached to a horizontal spring. Hence deduce an expression for its time period.	2
8.	Derive an expression for the time period of a body when it executes angular SHM	1
9.	What is damping? On what factors the damping depends?	1
10.	What are damped vibrations? Establish the differential equation of motion for a damped harmonic oscillator and obtain an expression for displacement. Discuss the case of heavy damping, critical damping and low damping.	2
11.	What do you mean by forced harmonic vibrations? Discuss the vibrations of a system executing simple harmonic motion when subjected to an external force.	2
12.	What is driven harmonic oscillator? How does it differ from simple and damped harmonic oscillator?	2
13.	What is resonance? Explain the sharpness of resonance.	2
14.	Illustrate an example to show that resonance is disastrous sometimes.	2

SI No	**Numerical Problems**	**CO**

1.	A cantilever of rectangular cross section has depth 3 mm and breadth 2 cm has the same restoring moment as that of a circular cross section of the same material and half the radius of curvature. Calculate the radius of the circular cantilever. **Solution:** Restoring moment = YI_g/R. $I_{g,}$ circle= $\pi r^4/4$, $I_{g,\ rectangle} = bd^3/12$. $R_{circle} = 0.5\ R_{rectangle}$. Thus we have $I_{g,}$ circle/ $R_{circle} = I_{g,\ rectangle} / R_{rectangle}$ So, $R \times \pi r^4/4 = 2R \times bd^3/12$. So $r^4 = (2/3\pi)\ bd^3$. $r = \{(2/3\pi)\ bd^3\}^{1/4}$	**3**
2.	Calculate the tensile strain of the top layer of a cantilever of thickness 2 mm and Radius of curvature = 1 m. **Solution:** Strain = r/R = 1mm/1m = 0.001	**3**
3.	Calculate the shearing modulus of a cubic block of size 20 cm, whose top layer displaces by 1 mm under the action of a tangential force of 10 Mega – Newton. **Solution:** Strain = 1 mm/20 cm = 0.005. Shearing Modulus = (Force/Area)/Strain $= (10^7/(400 \times 10^{-4}))/(5 \times 10^{-3}) = 5 \times 10^{10} N/m^2$	**3**
4.	The aspect ratio (r/L) of a cylindrical wire is x. Calculate the strain on this wire if it is twisted by an angle θ. Ans : Strain = (r/L) θ	
5.	How much should the thickness of a wire be increased, such that the force required to twist it becomes double ? Solution : The couple per unit has to become double. $C = \pi\eta r^4/2L$. Thus if C has to become double, $r^4 = 2$. Thus r = 1.1892	
6.	Estimate the tensile stress acting on the top surface of a long rectangular cross section cantilever of thickness t, given its radius of curvature is R and Young's Modulus is Y. Solution : Distance of the top layer from the neutral layer = t/2. So strain of the top layer = (t/2)/R = (t/2R) Tensile stress acting on the top layer = Yt/2R	
7.	A particle executes SHM of period 31.4 second and amplitude 5cm. Calculate its maximum velocity and maximum acceleration. **Solution:** The maximum velocity at y =0 in $v = \omega\sqrt{A^2 - y^2}$, $\therefore v_{max} = \omega A$ $\omega = \frac{2\pi}{T} = \frac{2\pi}{31.4} = 0.2 radian$ $\therefore v_{max} = 0.2 \times 5 = 1.0 cm/\sec = 0.01 m/s$ At the maximum displacement, i.e., at the extreme position x=A, maximum acceleration is $-\omega^2 A$ $a = -0.2^2 \times 5 = -0.002 m/s^2$	3
8.	A circular plate of mass 4kg and diameter 0.10 metre is suspended by a wire which passes through its centre. Find the period of angular oscillations for small displacement if the torque required per unit twist of the wire is 4×10^{-3} N-m/radian.	3

	Solution: **Moment of Inertia** $I=\frac{MR^2}{2}=\frac{4(0.05)^2}{2}=0.005kgm^2$ **Time period is given by** $T=2\pi\sqrt{\frac{I}{C}}==2\times3.14\times\sqrt{\frac{0.005}{4x10^{-3}}}$ **=7.021s.**	
9.	A mass of 6kg stretches a spring 0.3m from its equilibrium position. The mass is removed and another body of mass 1kg is hanged from the same spring. What would be the period of motion if the spring is now stretched and released? **Solution:** $F=ky,\ k=\frac{F}{y}=\frac{mg}{y}=\frac{6\times9.8}{0.3}=196N/m$ and $T=2\pi\sqrt{\frac{m}{k}}=2\times3.14\sqrt{\frac{1}{196}}=0.45s$	3
10.	A vibrating system of natural frequency 500 cycles/sec, is forced to vibrate with a periodic force/unit mass of amplitude 100 x 10^{-5} N/kg in the presence of a damping/ unit mass of 0.01 x 10^{-3} rad/s. Calculate the maximum amplitude of vibration of the system. **Given** :Natural frequency = 500 cycles/sec, Amplitude of the force / unit mass , Fo/m=100 x 10^{-5} N/kg, Damping coefficient , r/m = 0.01 x 10^{-3} rad/s **Solution:** Maximum amplitude of vibration $A=\frac{f}{2b\omega}=\frac{F_o/m}{(r/m)\omega}$ $\therefore A=\frac{100\times10^{-5}}{0.01\times10^{-3}\times2\times\pi\times500}=0.0318m.$ $\therefore$The maximum amplitude of vibration of the system is 0.0318meter	3
11.	A circuit has an inductance of $1/\pi$ henry and resistance 100Ω. An A. C. supply of 50 cycles is applied to it. Calculate the reactance and impedance offered by the circuit. **Solution:** The inductive reactance is $X_L=\omega_d L=2\pi nL$ Here $\omega_d=2\pi n=2\pi$ x 50= 100π rad/sec and $L=1/\pi$ Henry. $X_L=\omega_d L=2\pi nL=2\pi$ x 50x $1/\pi$ =100Ω. The impedance is $Z=\sqrt{R^2+X_L^2}=\sqrt{100^2+100^2}=141.4\Omega$	3
12.	A series LCR circuit has L=1mH, C=0.1μF and R=10Ω.Calculate the resonant frequency of the circuit. **Solution:** The resonant angular frequency of the circuit is given by $\omega=\frac{1}{\sqrt{LC}}$ Here L=1mH and C=0.1μF $\omega=\frac{1}{\sqrt{10^{-3}x10^{-7}}}=10^5 rad/s$	3

Reference Books	
1	Oscillations and Waves, Satya Prakash, Pragati Prakshan, third edition, 2005.
2	Engineering Physics Hitendra K Malik and A K Singh, Tata McGraw Hill Education Private Limited, 2009, ISBN: 978-0-07-067153-9.
3	A Text book of Engineering Physics Dr. M N Avadhanulu, Dr. P. G. Kshirsagar, S. Chand & Company Private limited. Revised edition 2015.
4	Engineering Physics R K Gaur and S L Gupta, Dhanpat Rai Publications, Revised edition 2011.